

FRACTIONAL BROWNIAN MOTION AND ASYMPTOTIC BAYESIAN ESTIMATION

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ABSTRACT. In this paper, we study the recovery of the Hurst parameter from a given discrete sample of fractional Brownian motion with statistical inverse theory. In particular, we show that in the limit the posteriori distribution of the parameter given the sample determines the parameter uniquely. In order to obtain this result, we first prove various strong laws of large numbers related to the problem at hand and then employ these limit theorems to verify directly the limiting behaviour of posteriori distributions without making additional technical or simplifying assumptions that are commonly used.

1. INTRODUCTION

We study the recovery of the Hurst parameter from a given discrete sample of fractional Brownian motion with statistical inverse theory. In particular, we show that in the limit the posteriori distribution of the parameter given the sample determines the parameter uniquely.

Fractional Brownian motion Z^H is a one parameter generalization of the standard Brownian motion B introduced in [22]. The generalization corresponds to changing the variance function $\mathbf{V}B_t = |t|$ to the variance function $\mathbf{V}Z_t = |t|^{2H}$ where the Hurst parameter $H \in (0, 1)$. It can be shown that with this choice the fractional Brownian motion exists as a stochastically continuous Gaussian process with stationary increments (c.f. e.g. [13]). These increments are not, however, independent unless $H = \frac{1}{2}$ which corresponds to the Brownian motion case. This makes the analysis of these processes more involved.

The inverse problem we have in mind is the usual parameter estimation problem. We sample a given signal $X^{\hat{H}}$ at equidistant time instances $t_j = j/n$ for every $j = 0, 1, \dots, n$. From this data we form the increments $Y_j^{\hat{H}} := X^{\hat{H}}(t_j) - X^{\hat{H}}(t_{j-1})$. The reason for using the increments is motivated by the stationarity.

We formulate the parameter estimation problem as the Bayesian estimation problem:

Determine the conditional probability distribution of H given the sample $(Y_1^{\hat{H}}, \dots, Y_n^{\hat{H}})$. From the conditional

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probability distribution construct estimators for the true parameter \hat{H} .

We solve this using the standard Bayesian methods with the assumption that the prior is noninformative. This leads to the solution of form

$$\mathbf{P}(H \in U \mid \xi_{\hat{H},n}) = C_n(\xi_{\hat{H},n}) \int_U \frac{n^{nu}}{\sqrt{|T_n(f_u)|}} e^{-\frac{1}{2}n^{2(u-\hat{H})}Q_n(\xi_{\hat{H},n},u)} du$$

where $T_n(f)$ is the $n \times n$ Toeplitz matrix corresponding to the symbol f , the $|\cdot|$ stands for the determinant and where Q_n is the quadratic form

$$Q_n(y, u) = \langle y, T_n(f_u)^{-1}y \rangle.$$

The data $\xi_{\hat{H},n}$ we use in the posteriori solution is the rescaled increments $\xi_{\hat{H},n} := n^{\hat{H}}(Y_1^{\hat{H}}, \dots, Y_n^{\hat{H}})$. The symbol f_u corresponds to the covariance operator of the increments of fractional Brownian motion Z^u sampled at integer points.

This solution provides a numerical reconstruction method. However, the computational cost of numerically calculating the quadratic form Q_n is both expensive and quite unstable. Moreover, the computation of the Toeplitz determinant is likewise rather expensive and unstable. Therefore, only relatively small values of n can be used in numerical analysis and for small n the reconstruction from a simulated data does not appear to be very consistent with the true parameter value.

The estimation of the Hurst parameter of fractional Brownian motion from the measured data has a vast literature. It has been applied, for instance, to estimation in financial markets when we assume that there is a long term memory effects [1, 2, 29] and river flows [19, 24]. The name of the Hurst parameter comes from the study of River Nile by Hurst [19] in 1951.

The question we wanted to analyse is how these distributions behave asymptotically as the number n of samples tends to infinity.

This problem and related questions goes back at least 65 years and has an extensive literature. The first approach seems to be the line of estimation which could be called the methods of *Whittle approximation* type. This was introduced by Whittle [32] in 1951 and it has been applied for many different kinds of approximations for the maximum a posteriori (MAP) estimate (c.f. [3, 7, 9, 33, 16, 18, 23, 25, 30]). This corresponds to replacing the inverse matrix $T_n(f_u)^{-1}$ by the Toeplitz matrix $T_n(1/f_u)$.

The first significant contribution to the estimation problem for fractional Brownian motion following this line of reasoning came from Taqqu and Fox [15]. In [15] Fox and Taqqu showed that the estimator corresponding to the Whittle approximation (without the determinant) is asymptotically normal and gives the true parameter value \hat{H} in the

limit *provided* that $\hat{H} > \frac{1}{2}$. Afterwards the estimation result of Fox and Taqqu has been used and sharpened by various authors (see for instance [4, 21]).

The limitation $\hat{H} > \frac{1}{2}$ has been present in most of these studies and it has only been removed by different kinds of estimators [5, 31]. This limitation comes from the techniques used. However, the more profound limitation is that of using the Whittle approximation. The authors have not found any results that would give reasons to believe that this approximation would be a good approximation for the original problem, even though it might be superior way for estimating the parameters numerically. The heuristics and referred articles in [4, 8] are only concerned of the inversion of Toeplitz matrices and not of the mapping properties of the perturbation caused by the using an approximation. During the preparation of this manuscript, the authors found that the perturbation is not a classical (compact) perturbation even in the correct scale of function spaces. This result is, however, omitted from this manuscript and will be part of a later study. Naturally, these approximations were done to make the numerics faster and they work nicely and even with finite band approximation (see [12]).

In the present article, we tackle the original problem without using any approximations. The non-approximative results for the MAP estimator have been considered by Dahlhaus ([4, 10, 11]), but as far as the authors are aware, the estimation for the whole posteriori distribution is not done before. We show that the posteriori distribution converge weakly to point mass on top of \hat{H} almost surely (Corollary 7.14). This follows from the characterization result for the asymptotic conditional distribution (Theorem 7.13). More precisely, we show that with probability one the conditional distribution of \tilde{H}_n is asymptotically standard normal with mean α_n where the random variable \tilde{H}_n is a rescaled version of the random variable H . As a part of this result we deduce that the asymptotic variance of H around the mean α_n is of order $n^{-1}(\log n)^{-2}$ (Lemma 7.10). Furthermore, we show that the means α_n converge to \hat{H} as $n \rightarrow \infty$ with a rate of order $(\log n)^{-2}$ (Lemma 7.6).

From this asymptotic normality we can read that the usual estimators like conditional mean and the MAP estimates are biased for large n but they are asymptotically unbiased and asymptotically exact.

In the proof of the asymptotic parameter estimation result we use the asymptotics for Toeplitz determinants with one Fisher–Hartwig singularity [14]. In order to handle the randomness coming from the random quadratic form, we prove the (uniform) *Strong Law of Large Numbers* (SLLN) for the quadratic form Q_n appearing in the posteriori distribution. This uniform SLLN (Theorem 6.1) is the most involved part of the article since it builds upon the previous two Theorems (Theorem 3.1 and 4.1) and proving it requires different techniques from

probability theory, some ideas from the theory of Toeplitz operators and asymptotics for the inverse Toeplitz matrices [26, 27].

The rest of the paper is structured as follows: We start in Section 2 by briefly introducing our notation. In Section 3, we prove a generic strong law of large numbers for a sequence of Gaussian quadratic forms (Theorem 3.1). This result forms the basis for the rest of the asymptotic results. There are many special cases of the result in the literature, but as far as the authors are aware, the result of Theorem 3.1 is novel.

In Section 4, we show the main pointwise strong law of large numbers for the sequence of quadratic forms arising from the finite samples of fractional Brownian motion (Theorem 4.1). The main novelty is that with this result we obtain for which Hurst parameter values the random quadratic forms have almost sure limits and for what it diverges. This enables us to remove the usual technical restrictions of $H > \frac{1}{2}$. The proof relies on the earlier results of Rambour and Seghier [26, 27]. In our case, the Toeplitz symbol g_α (which we will introduce in Section 2) does not satisfy the assumptions of the main theorems in [26, 27] for every Hurst parameter value. Therefore, we have to generalize the results of [26, 27] for slightly larger class of symbols. To do so we deduce in the Section 5 few lemmata that provide the needed factorisation of the Fisher–Hartwig symbol. For these we follow the techniques of Grenander and Szegő [17]. Since the proofs are mainly technical lemmata, the proofs are postponed to the appendix.

Subsequently, in Section 6 we improve the pointwise strong law of large of the quadratic forms arising from the finite samples of fractional Brownian motion into a functional strong law of large numbers which we call as the Uniform Law of Large Numbers (Theorem 6.1). This result implies that posteriori distributions of the Hurst parameters given the finite samples of FBM converge weakly almost surely as the sample size grows to infinity. The proof relies on Helly’s Selection Theorem and analysis of the Fisher–Hartwig singularity of the symbols g_α together with the pointwise Strong Law of Large numbers (Theorem 4.1).

Subsequently, in Section 7, we use the Theorem 6.1 together with simple asymptotic analysis to derive the asymptotic behaviour of the sequence of posteriori distributions of the unknown Hurst parameter H given the finite samples of FBM (Theorem 7.13. Finally, the Appendix is divided into Sections A, B, C, D and E that consists the proofs of auxiliary lemmata of Sections 3, 4, 5, 6, and 7, respectively.

2. NOTATIONS

For the reason of notational compactness we use the Iverson brackets in this paper. Since it is a rather atypical notation in the field, we introduce it properly.

Notation (Iverson bracket). *The notation $[\cdot]$ is the Iverson bracket (see for example [20])*

$$[A] := \begin{cases} 1, & \text{if } A \text{ is true,} \\ 0, & \text{otherwise.} \end{cases}$$

We also use the Iverson brackets to denote the indicator functions by notation

$$[A](x) := [x \in A].$$

The benefit of this is that we can then use the standard trick of probability theory to eliminate the elementary events from expectations. For example, we can write $\mathbf{E}X[A]$ instead of the more cumbersome notations

$$\mathbf{E}X[\cdot \in A] = \int_{\Omega} X(\omega) [\omega \in A] \mathbb{P}(d\omega) = \int_A X(\omega) \mathbb{P}(d\omega).$$

Because of this, we will never use square brackets as an alternative to parenthesis. The only case, where we use square brackets and not mean the indicator function is when we denote closed intervals. However, these cases are easily recognised from the context.

Toeplitz matrices and operators have a significant role in this article. Toeplitz matrices and operators are in a close relation with circulant matrices and convolution operators. In this article, we mean by a convolution operator an infinite dimensional matrix index over integers that forms a Fourier pair with a multiplication by a *symbol* acting on smooth periodic functions on top of torus \mathbb{T} . We denote the convolution operator and the symbol as mappings $C(g_\alpha): c_{00} \rightarrow \mathbb{C}^{\mathbb{Z}}$,

$$\langle C(g_\alpha)\mathbf{e}^j, \mathbf{e}^k \rangle := c(g_\alpha)(j - k) := \int_{\mathbb{T}} e^{-i(j-k)t} g_\alpha(t) dt$$

where g_α is the *symbol* or *spectral density function* and c_{00} stands for the sequences with only finitely many nonzero elements. The importance of these convolution operators stem from the spectral representation for the fractional Brownian noise. Yakov G. Sinaï has shown in 1976 [28] the following fact.

Lemma 2.1. *The spectral density function of Fractional Brownian noise has a representation*

$$(1) \quad f_H(\lambda) = C_H |e^{i\lambda} - 1|^2 \sum_{k \in \mathbb{Z}} |\lambda - 2\pi k|^{-(2H+1)},$$

where $C_H \in \mathbb{R}$ is a norming constant.

Proof. See [28]. □

For symmetry reason we use $g_\alpha := f_{\frac{1}{2}+\alpha}$ with $-\frac{1}{2} < \alpha < \frac{1}{2}$. This is since by Sinaï's result we have

$$\begin{aligned} \forall \alpha \in (0, 1): f_\alpha(t) [|t| < \varepsilon] &= c_\alpha t^2 (1 + \mathcal{O}(t^2)) (|t|^{-(2\alpha+1)} + \mathcal{O}(1)) [|t| < \varepsilon] \\ &= c_\alpha |t|^{1-2\alpha} (1 + o(t)) [|t| < \varepsilon]. \end{aligned}$$

Rewriting this for the new symbol yields

$$\begin{aligned} \forall \alpha \in (-\tfrac{1}{2}, \tfrac{1}{2}): g_\alpha(t) [|t| < \varepsilon] &= f_{\frac{1}{2}+\alpha}(t) [|t| < \varepsilon] \\ &\asymp |t|^{-2\alpha} [|t| < \varepsilon], \end{aligned}$$

which reveals the simple relation between the symbol g_α and the order of the zero at origin.

Usually, the we only choose some part of the convolution operator $C(f)$ and these are the Toeplitz operators $T(f)$ and Toeplitz matrices $T_n(f)$. In this work, we don't really need the infinitedimensional operators $C(f)$ and $T(f)$, since the mapping properties are not nice enough in this setting, so we only really need the Toeplitz matrices T_n which are defined as

$$T_n(g_\alpha) = (C(g_\alpha)_{jk})_{j,k=1}^n = (c(g_\alpha)(j-k))_{j,k=1}^n$$

These Toeplitz operators and matrices correspond to the convolution operator acting on analytic functions projected to the analytic functions. The mappings that correspond to the convolution operator acting on analytic function but projected to the anti-analytic functions are called Hankel operators and Hankel matrices. While these kind of mappings are really needed in this work, we, however, only need them implicitly (see Lemma 4.3) and therefore, we don't give the actual definitions.

In several occasions, we need to represent a function in a point free manner, so we use notation $f = x \mapsto f(x)$ to denote that f is a mapping and $f(x)$ is its value. For a curried function $f = x \mapsto (y \mapsto F(x, y))$, the $f(x) = y \mapsto F(x, y)$ and $f(x)(y) = F(x, y)$. When we don't need to give a name for a function, we use $x \mapsto \dots$ to denote the anonymous function.

In Section 3 and especially in Section A, we use extensively Frobenius norm, Frobenius inner product and tensor products of matrices. We denote the Frobenius inner product of two matrices A and B by $A : B$ and this is defined as

$$A : B = \sum_{jk} A_{jk} B_{jk} = \text{Tr}(AB^\top)$$

The Frobenius norm of a square matrix A is $\|A\|_F = \sqrt{A : A}$. We interpret multi-indices as words, i.e. for a matrix $A = (A_{jk})_{jk}$ we interpret $\rho = jk$ as a word of two letters. The words ρ and η can be concatenated

with concatenation operation $\rho\&\eta$ which is a word (or multi-index) obtained by joining the two words. For example, if $\rho = jk$ and $\eta = lm$ then $\rho\&\eta = jklm$.

We will use tensor notation to deal with multilinear objects that we arrive when computing higher moments and we identify vectors and matrices with tensors with one letter words and tensors with two letter words, respectively. In this work we will mean by (covariant) tensors the mappings from words (i.e. elements of $\mathbb{N}^* = \mathbb{N} \cup \mathbb{N}^2 \cup \dots$) to scalar field, i.e. $A = \rho \mapsto A_\rho$. The tensor product of two tensors A and B is defined as $A \otimes B = (\rho\&\eta) \mapsto A_\rho B_\eta$. The tensor power $A^{\otimes n}$ is the n -fold tensor product $A \otimes \dots \otimes A$.

In Section A and for instance in the claim of Lemma 3.3, we denote the falling product by $a^{\underline{n}}$. The falling product is defined as $a^{\underline{n}} = a(a-1)\dots(a-n+1)$. In Section 3 we use for the first time the lattice operations \wedge and \vee to denote the minimum and maximum, respectively. The multi-index power means the usual $x^\rho = x_1^{\rho_1} x_2^{\rho_2} \dots$ where the vector x and the multi-index ρ share the same finite dimension.

Throughout the work, we denote majorization by $f \lesssim g$, by which we mean that there is a positive constant $c > 0$ such that $f \leq cg$. Similarly, $f \asymp g$ is $f \lesssim g \lesssim f$. We typically write these in a pointed manner, i.e. as $f(x) \lesssim g(x)$ and by context it should be clear which argument we are considering. Moreover, the domain where this majorization is concerned is usually some neighbourhood of some infinity point, but this should be clear from the context. Few times we denote $f \ll g$ instead of $f = o(g)$.

We will use ellipsis (i.e. "...") to denote something that we decided to temporarily omit writing. This is typically used together with integration where we temporarily don't write the integrand explicitly. Furthermore, when we write

$$\int_{\mathbb{T}} \dots dt$$

we mean integration with respect to the normalized Lebesgue measure on a torus $\mathbb{T} = [-\pi, \pi)$. We will call the interval $[-\pi, \pi)$ as a torus even though we don't explicitly map it to a unit circle on a plane.

3. STRONG LAW OF LARGE NUMBERS FOR SYMMETRIC GAUSSIAN QUADRATIC FORMS

In this section we prove an auxiliary limit result that we need for the later limit theorems. We will denote by (A_n) a sequence of symmetric matrices in $\mathbb{R}^{n \times n}$ and assume that (ξ_n) is a sequence of Gaussian random variables with zero mean and covariance matrices $C_n \in \mathbb{R}^{n \times n}$.

Theorem 3.1. *Suppose $0 \leq \gamma < 1$. If $\left\| C_n^{1/2} A_n C_n^{1/2} \right\|_F \lesssim n^\gamma$, then*

$$\lim_{n \rightarrow \infty} n^{-1} (\langle \xi_n, A_n \xi_n \rangle - \mathbf{E} \langle \xi_n, A_n \xi_n \rangle) = 0$$

almost surely.

In the latter part of this section, we will fix n and therefore, we will drop the subscript n to simplify notations and to release n for other uses. We will denote

$$(2) \quad \Theta(B, n) = \sum_{\rho} B \otimes A_{\rho}^{\otimes(n-1)} \mathbf{E} \xi_{\rho}^{\otimes 2n} \quad \text{and} \quad \Theta(n) = \Theta(A, n).$$

We note that $\Theta(n) = \mathbf{E} \langle \xi, A\xi \rangle^n$. Furthermore, we will denote

$$(3) \quad R_j = \text{Tr}(AC)^j \quad \text{and} \quad R^{\mathbf{k}} = \prod_j R_{\mathbf{k}_j}$$

for every $\mathbf{k} \in \mathbb{N}_+^*$. We will need to have a control for the multi-indices \mathbf{k} so we define few sets of multi-indices. First we need to know the number of ones in a multi-index. We denote the counting function by θ , and the cumulative functions by s_k

$$(4) \quad \theta(\mathbf{k}) = \sum_j [\mathbf{k}_j = 1] \quad \text{and} \quad s_k(\mathbf{k}) = \sum_j \mathbf{k}_j [1 \leq j \leq k].$$

With the counting and cumulative functions we define

$$(5) \quad \begin{aligned} J(m, n) &= \{\mathbf{k} \in \mathbb{N}_+^m; s_m(\mathbf{k}) = n\} \quad \text{and} \\ J_l(m, n) &= \{\mathbf{k} \in J(m, n); \theta(\mathbf{k}) = l\}. \end{aligned}$$

To each multi-index $\mathbf{k} \in J_0(n, m)$ we associate the following number

$$(6) \quad \log c(\mathbf{k}) = - \sum_{j=2}^n \log(s_n(\mathbf{k}) - s_{j-1}(\mathbf{k})) + \sum_{j=1}^{s_n(\mathbf{k})-1} \log j$$

We still need one auxiliary function so that we can formulate the representation lemma for the cumulant functions. We denote the cumulant function by Ψ

$$(7) \quad \Psi(N) = \sum_{n=0}^N \binom{N}{n} (-1)^{N-n} \Theta(n) R_1^{N-n}.$$

We can now formulate the representation result.

Lemma 3.2. *For $N > 0$ we have*

$$\Psi(N) = \sum_{m=1}^N 2^{N-m} \sum_{\mathbf{k}} [\mathbf{k} \in J_0(m, N)] R^{\mathbf{k}} c(\mathbf{k})$$

This representation has two important aspects that are that each multi-index that appears on the right-hand side has $s(\mathbf{k}) = N$ and $\theta(\mathbf{k}) = 0$. The proof of Lemma 3.2 is given in the end of this section.

We will start by proving the strong law.

Proof of Theorem 3.1. According to the assumption, the values $R_j = R_j(n)$ satisfy

$$R_{2j}(n) \leq \|C_n^{1/2} A_n C_n^{1/2}\|_F^{2j} \lesssim n^{2j\gamma}$$

and by Cauchy–Schwarz inequality for Frobenius inner product

$$R_{2j+1}(n) \leq \|C_n^{1/2} A_n C_n^{1/2}\|_F^{2j} \|C_n^{1/2} A_n C_n^{1/2}\|_F \lesssim n^{(2j+1)\gamma}.$$

Let us denote

$$X_n = \langle \xi_n, A_n \xi_n \rangle - \mathbf{E} \langle \xi_n, A_n \xi_n \rangle.$$

The estimates for $R_j(n)$ combined with Lemma 3.2 gives

$$\mathbf{E} X_n^{2N} \lesssim n^{2N\gamma}.$$

Therefore, if we choose $N > (1 - \gamma)^{-1}$ we see that

$$\mathbf{E}(n^{-1} X_n)^{2N} \lesssim n^{-2}.$$

This estimate together with application of Chebysev Inequality implies that

$$\sum_n \mathbf{P}(n^{-1} |X_n| > \varepsilon) < \infty$$

for every $\varepsilon > 0$. The claim of the Theorem follows immediately from this by Borel–Cantelli Lemma. \square

Remark. The proof of Theorem 3.1 relies heavily on the representation given by Lemma 3.2. It is straightforward to show a similar representation but without the extra condition $\theta(\mathbf{k}) = 0$ for every \mathbf{k} appearing on the right. If we suppose that $\lim n^{-1} \mathbf{E} \langle \xi_n, A_n \xi_n \rangle = \lim n^{-1} R_1(n) = F(A, C) > 0$, then terms $R_1(n) \asymp n$. In worst case terms with $\theta(\mathbf{k}) \asymp N$ would prevent obtaining the convergence result for γ sufficiently close to 1.

We first compute the first representation formula for $\Theta(n)$ which is a simple recursive formula.

Lemma 3.3. *We have for $n \geq 1$*

$$\Theta(n) = \sum_{j=1}^n 2^{j-1} (n-1)^{j-1} R_j \Theta(n-j)$$

Using this we obtain the first exact representation. For this we denote

$$(8) \quad \Theta(n) = \sum_{j=k}^n a_k(j, n) \Theta(n-j)$$

for every $k \leq n$. We note that a_1 is defined by Lemma 3.3, moreover applying the same lemma gives recursive formula for a_k .

Lemma 3.4. *We have for $1 \leq k < j \leq n$ that*

$$a_{k+1}(j, n) = a_k(k, n) a_1(j-k, n-k) + a_k(j, n).$$

Since $\Theta(n) = a_n(n, n)$, solving the recursion equation for a_k solves Θ as well.

Lemma 3.5. *We have for $n \geq 1$ that*

$$\Theta(n) = \sum_{m=1}^n \sum_{\mathbf{k} \in J(m, n)} 2^{n-m} R^{\mathbf{k}} \prod_{j=1}^m (n - s_{j-1}(\mathbf{k}) - 1)^{\underline{\mathbf{k}_j - 1}}$$

The value $R^{\mathbf{k}}$ is invariant with respect to permutations of \mathbf{k} . Furthermore, we note that whenever $\mathbf{k}_j = 1$ the term satisfies

$$(n - s_{j-1}(\mathbf{k}) - 1)^{\underline{\mathbf{k}_j - 1}} = 1.$$

Moreover, $J(m, n) = \bigcup_l J_l(m, n)$ where $J_0(m, n)$ will be called the good part and the rest as the bad part. The goal is to show that in the end the bad part cancels out, so we need to have explicit division in to these two parts. For this we introduce a new set of multi-indices

$$L(m, n) = \{1, \dots, n\}^m \cap \text{INC}$$

where INC denotes strictly increasing sequences of any length. In this way we can divide $J(m, n)$ in to two parts, to part consisting of 1's and the rest. Therefore, we denote for every $\mathbf{k} \in J_0(m-l, n-l)$ and every $\lambda \in L(m-l, m)$

$$(9) \quad \pi_j(\mathbf{k}, \lambda) = 1 + \sum_l (\mathbf{k}_l - 1) [\lambda_l = j].$$

We note that $\pi(\mathbf{k}, \lambda) \in J_l(m, n)$ and every element in $J_l(m, n)$ is obtained in the process.

The last remaining part is the auxiliary function is

$$(10) \quad \Lambda(n, \mathbf{k}) = \sum_{\lambda \in L(m-l, m)} \prod_{j=1}^{m-l} (n - s_{\lambda_j-1}(\pi(\mathbf{k}, \lambda)) - 1)^{\underline{\mathbf{k}_j - 1}}$$

whenever $\mathbf{k} \in J_0(m-l, n-l)$. With the help of these notation we can give the next representation for Θ .

Lemma 3.6. *We have for $n \geq 1$ that*

$$\Theta(n) = \sum_{m=1}^n \sum_{l=0}^m 2^{n-m} R_1^l \sum_{\mathbf{k}} R^{\mathbf{k}} \Lambda(n, \mathbf{k}) [\mathbf{k} \in J_0(m-l, n-l)].$$

It turns out that there is a simple representation for Λ .

Lemma 3.7. *We have for every $n \geq 1$ and $\mathbf{k} \in J_0(m, n-l)$ that*

$$\Lambda(n, \mathbf{k}) = c(\mathbf{k}) \binom{n}{n-l}.$$

We can now prove the main representation lemma (Lemma 3.2).

Proof of Lemma 3.2. Combining Lemmata 3.6 and 3.7 we have by using little algebra that when $n \geq 0$

$$\begin{aligned} R_1^{N-n}\Theta(n) &= R_1^N + \sum_{M=1}^n R_1^{N-M} \sum_{m \geq 1} 2^{M-m} \sum_{\mathbf{k} \in J_0(m, M)} R^{\mathbf{k}} c(\mathbf{k}) \binom{n}{M} \\ &= \sum_{M=0}^n R_1^{N-M} \kappa(M) \binom{n}{M} \end{aligned}$$

Since $\Theta(0) = 1$ we obtain that

$$\Psi(N) = \sum_{M=0}^N \kappa(M) \sum_{n=M}^N \binom{N}{n} (-1)^{N-n} \binom{n}{M} = \kappa(N)$$

and the claim follows. \square

4. POINTWISE STRONG LAW OF LARGE NUMBERS FOR FBN

We will apply the Theorem 3.1 in order to obtain a pointwise SLLN for FBN.

Theorem 4.1. *Suppose $\alpha_- + \beta_+ < \frac{1}{2}$. Suppose $\xi_n \sim N(0, T_n(g_\beta))$ for every n . Then*

$$\lim_{n \rightarrow \infty} n^{-1} \langle \xi_n, T_n(g_\alpha)^{-1} \xi_n \rangle = \int_{\mathbb{T}} \frac{g_\beta(t)}{g_\alpha(t)} dt$$

almost surely.

For this we need few propositions and couple of lemmata. We will postpone the proofs of these results to the Appendix (Section B). First proposition states that the Theorem 3.1 is applicable in our case.

Proposition 4.2. *For every $\alpha, \beta \in (-\frac{1}{2}, \frac{1}{2})$ we have*

$$\|T_n(g_\beta)^{1/2} T_n(g_\alpha)^{-1} T_n(g_\beta)^{1/2}\|_F \asymp n^{2(\alpha_- + \beta_+) \vee 1/2}$$

Next we need to express the inverse Toeplitz matrix as a perturbation of a Toeplitz matrix.

Lemma 4.3. *For every $\alpha \in (-\frac{1}{2}, \frac{1}{2})$ there exists a matrix $K_n(\alpha)$ such that*

$$T_n(g_\alpha)^{-1} = T_n(g_\alpha^{-1}) + \frac{1}{2} (T_n(g_\alpha)^{-1} K_n(\alpha) + K_n(\alpha)^* T_n(g_\alpha)^{-1})$$

Let us denote $\tilde{K}_n(\alpha) = \frac{1}{2} (|K_n(\alpha)| + |K_n^*(\alpha)|)$. We note that when $\alpha = 0$, the matrix $\tilde{K}_n(0) = 0$. For the matrix $\tilde{K}_n(\alpha)$ we state the following properties.

Proposition 4.4. *For every $\alpha \in (-\frac{1}{2}, \frac{1}{2})$*

$$|T_n(g_\alpha)^{-1} \tilde{K}_n(\alpha)| \lesssim [\alpha \neq 0] n^{2\alpha_-} \tilde{K}_n(\alpha)$$

Furthermore, we still need one estimate

Proposition 4.5. *For every $\alpha \in (-\frac{1}{2}, \frac{1}{2})$ and $\beta \in (-\frac{1}{2}, \frac{1}{2})$ we have*

$$|T_n(g_\beta)| : \tilde{K}_n(\alpha) \lesssim [\alpha \neq 0] (n^{2\beta} \vee \log n)$$

These propositions are essential pieces for proving the SLLN for fractional Brownian noise.

Proof of Theorem 4.1. Let us denote $A_n = T_n(g_\alpha)^{-1}$, $B_n = T_n(g_\alpha^{-1})$ and $C_n = T_n(g_\beta)$. Furthermore, we will drop α from $K_n(\alpha)$ and $\tilde{K}_n(\alpha)$ since α is fixed. The assumption together with Proposition 4.2 implies that the Frobenius norm $\|C_n^{1/2} A_n C_n^{1/2}\|_F$ satisfies the requirements of the Theorem 3.1. Therefore,

$$\lim_{n \rightarrow \infty} (n^{-1} \langle \xi_n, A_n \xi_n \rangle - n^{-1} \mathbf{E} \langle \xi_n, A_n \xi_n \rangle) = 0$$

almost surely. We have for every n by Lemma 4.3 that

$$\mathbf{E} \langle \xi_n, A_n \xi_n \rangle = A_n : C_n = B_n : C_n + \frac{1}{2} (A_n K_n + K_n^* A_n) : C_n.$$

By Propositions 4.4 and 4.5 we have

$$\begin{aligned} |A_n K_n + K_n^* A_n| : |C_n| &\lesssim n^{2\alpha-} \tilde{K}_n : |C_n| \\ &\lesssim n^{2(\alpha- + \beta+)} \log n. \end{aligned}$$

This implies that

$$\lim_{n \rightarrow \infty} n^{-1} (\mathbf{E} \langle \xi_n, A_n \xi_n \rangle - B_n : C_n) = 0$$

On the other hand

$$n^{-1} B_n : C_n = \int_{\mathbb{T}} dt \int_{\mathbb{T}} ds \frac{g_\beta(t)}{g_\alpha(s)} h_n(t-s)$$

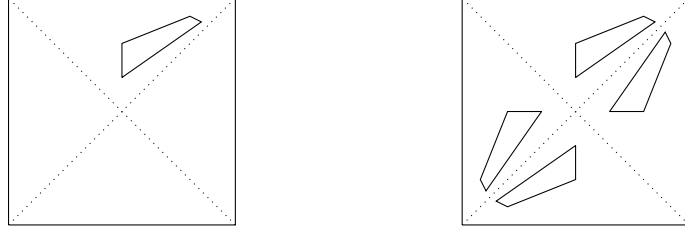
where h_n is the Fejér kernel. Therefore, the limit exists and equals the claimed value provided $g_\beta/g_\alpha \in L^1$. This, however, is equivalent with $\alpha - \beta > -\frac{1}{2}$ which follows from the assumption. \square

The proofs of Propositions 4.2, 4.4 and 4.5 and proof of Lemma 4.3 are postponed to Section B, as mentioned before. In the end of this section, we describe what is needed in order to obtain these and introduce two lemmata that cover the key points.

First, we consider some properties of the matrices $\tilde{K}_n(\alpha)$ in more detail. We note that $K_n(\alpha)$ is a sum of two products of two Hankel operators corresponding to symbols g_α and g_α^{-1} . If both symbols were bounded, the standard symbol calculus methods could be used, but in this case the properties has to be computed by hand.

We will denote

$$(11) \quad k_\alpha(x, y) = \frac{(x \vee y)^{-1+2|\alpha|}}{(x \wedge y)^{2|\alpha|}} + \frac{(1 - x \wedge y)^{-1+2|\alpha|}}{(1 - x \vee y)^{2|\alpha|}}.$$

FIGURE 1. Illustration of $\mathcal{S}(f)$

Lemma 4.6. *For $0 < |\alpha| < \frac{1}{2}$, we have that*

$$\tilde{K}_n(\alpha)_{ij} \lesssim n^{-1} k_\alpha(x, y)$$

where $nx = (i \vee 1) \wedge (n - 1)$ and $ny = (j \vee 1) \wedge (n - 1)$.

Propositions 4.2 and 4.4 require knowledge of the inverse matrix T_n^{-1} . For this we adapt the results of Rambour and Seghier [26, 27]. Let us introduce some notations. We will denote for every $x \in [0, 1]$

$$(12) \quad \tilde{x} = 1 - x \quad \text{and} \quad |x|_n = |x| \vee n^{-1}$$

With these notations can define

$$(13) \quad \begin{aligned} \mathcal{S}(f)(x, y) = & f(x, y) [y \geq (x \vee \tilde{x})] + f(y, x) [\tilde{x} \leq y < x] \\ & + f(\tilde{y}, \tilde{x}) [x \leq y < \tilde{x}] + f(\tilde{x}, \tilde{y}) [y < (x \wedge \tilde{x})]. \end{aligned}$$

Therefore $\mathcal{S}(f)$ is an extension of function f defined in a triangle $x \vee \tilde{x} \leq y \leq 1$ which is symmetric and invariant with respect to transformation $(x, y) \leftrightarrow (\tilde{x}, \tilde{y})$. We denote

$$(14) \quad I_d(x) = \{(x \vee \tilde{x}) \leq y < \tfrac{1}{2}(x + 1)\}, \quad I_b(x) = \{y \geq (\tilde{x} \vee \tfrac{1}{2}(x + 1))\}$$

and furthermore,

$$(15) \quad E_1^{(\alpha, n)}(x, y) = |y - x|_n^{-1+2\alpha} [y \in I_d(x)]$$

and

$$(16) \quad E_2^{(\alpha)}(x, y) = (y - x)^{-1+\alpha} x^\alpha \tilde{y}^\alpha [y \in I_b(x)]$$

These notations allow us to adapt the results for the elementwise asymptotics for the inverse matrices of Toeplitz matrices $T_n(g_\alpha)$.

Lemma 4.7. *When $0 < |\alpha| < \frac{1}{2}$ and we have*

$$|T_n(g_{-\alpha})_{ij}^{-1}| \asymp [i = j] + [i \neq j] n^{-1+2\alpha} \mathcal{S}(E_1^{(\alpha, n)} + E_2^{(\alpha)})(x, y)$$

when $xn = i \vee N \wedge (n - N)$, $yn = j \wedge (n - N)$.

The adaptation is, however, not entriy straightforward so we need further analysis for obtaining this lemma. This is done in the following section (Section 5).

5. FACTORISATION OF THE SYMBOL g_α

The proof of Lemma 4.7 is mostly technical and it relies on the asymptotic representation of Rambour and Seghier [26, 27]. In these articles, they obtain elementwise asymptotic representations of inverses of Toeplitz matrices with a single Fisher–Hartwig singularity. More precisely, they give their results to symbols of form $f_1\theta_{2\alpha}$ where f_1 is sufficiently smooth positive function (*a smooth perturbation*) and the $\theta_{2\alpha}$ is the pure Fisher–Hartwig singularity

$$(17) \quad \theta_{2\alpha}(t) = 2^\alpha(1 - \cos t)^\alpha.$$

Since we know already that $g_\alpha \asymp \theta_{-2\alpha}$ we in principle only have to show that $g_\alpha\theta_{2\alpha}$ is sufficiently smooth. In [26, 27], the assumption for smooth perturbation f is that $(\widehat{f_1}(k)k^{3/2}) \in \ell^1$, which is valid in our case only for $\alpha > -1/4$. Therefore in order to handle the case $-1/2 < \alpha \leq -1/4$ we have to improve their result.

Analysing the proofs of the main results in [26, 27] we observe that the symbol g_α only needs to satisfies the following conditions:

- immediate conditions: $g_\alpha \geq 0$, $g_\alpha \in L^1$ and $g_\alpha^{-1} \in L^1$
- $\log g_\alpha \in L^1$ (follows from previous, since $|\log x| \leq x + x^{-1}$).
- there exists a $q_\alpha \in H^2(\mathbb{T})$, a boundary trace of an analytic square integrable function, that satisfies $q_\alpha \overline{q_\alpha} = g_\alpha^{-1}$ and

$$(18) \quad C_\alpha \widehat{w}_\alpha(k) = \widehat{q}_\alpha(k) + o(k^{-\alpha-1})$$

where $w_\alpha(t) = (1 - e^{it})^\alpha$ and $C_\alpha = \lim_{t \rightarrow 0} q_\alpha w_{-\alpha}(t)$.

The last condition means that we factorize the symbol g_α into the product of an analytic and anti-analytic square root. We have a trivial factorisation for the pure Fisher–Hartwig singularity $w_\alpha \overline{w}_\alpha = \theta_{2\alpha}$. The condition then states that the Fourier coefficients of the analytic square root coincide with the Fourier coefficients of the analytic square root of the pure Fisher–Hartwig singularity asymptotically and upto a constant.

Following Grenander–Szegő [17] we have a clear recipe for this factorisation (of $f \geq 0$ defined on $\partial\mathbb{D} \sim \mathbb{T}$, say)

- let u be the harmonic extension of $1/2 \log f$ in \mathbb{D}
- let v be the harmonic conjugate of u with $v(0) = 0$
- the required analytic square root of f is then the boundary trace of $\exp(F)$, where $F = u + iv$.

This *Riemann–Hilbert problem* has a unique solution if $f \in L^1$ and $\log f \in L^1$.

Lemma 5.1. *When $f \in L^1(\mathbb{T})$, $f \geq 0$ and $\log f \in L^1(\mathbb{T})$, then the function q given by*

$$(19) \quad q = \sqrt{f} \exp(i/2 \mathcal{H}_0(\log f))$$

satisfies

$$(20) \quad q \in H^2(\mathbb{T}) \quad \text{and} \quad f = q\bar{q}$$

where \mathcal{H}_0 is the Hilbert transform on the torus

$$\mathcal{H}_0 f(x) = p.v. \int_{\mathbb{T}} \cot\left(\frac{t-x}{2}\right) f(t) dt$$

Moreover, a function q satisfying (20) is unique upto a multiplication with an inner function.

In the sequel we will denote

$$(21) \quad \psi_\alpha^{-2} = g_\alpha \theta_{2\alpha}$$

the perturbation of the pure Fisher–Hartwig symbol that we need to obtain the FBN symbol g_α . We will denote by

$$(22) \quad r_\alpha = \psi_\alpha \exp\left(i\mathcal{H}_0(\log \psi_\alpha)\right)$$

the analytic square root of ψ_α^2 given by Lemma 5.1. We have

Lemma 5.2. *For every $0 < |\alpha| < 1/2$ there exists an $q_\alpha \in H^2(\mathbb{T})$ such that $q_\alpha \overline{q_\alpha} = 1/g_\alpha$ and for almost every $t \in \mathbb{T}$ it holds that*

$$q_\alpha(t) = w_\alpha(t) r_\alpha(t).$$

This representation is enough for showing the required estimate for the Fourier coefficients.

Lemma 5.3. *For every $0 < |\alpha| < 1/2$ and for every $k \geq 1$ we have*

$$\widehat{q}_\alpha(k) = C_\alpha \widehat{w}_\alpha(k) + \mathcal{O}k^{-2-\alpha}.$$

Proof of Lemma 5.3. The result follows by combining Lemmata 5.2 and 5.6 since the Fourier transform of the product is the convolution of the Fourier transforms. \square

This result implies that we may use the elementwise results of Rambour and Seghier since g_α satisfies the condition (18), even though the perturbation ψ_α^2 is not always as smooth as they required (see Lemma 5.4).

The estimate for the convolution of Fourier transforms (Lemma 5.6) follows from the following lemmata.

Lemma 5.4. *Let $u = \log \psi_\alpha$. Then we have the following asymptotic estimates for the Fourier coefficients*

$$(23) \quad |\widehat{r}_\alpha(k)| = |\widehat{u}(k)| \asymp \left| \widehat{\psi}_\alpha(k) \right| \asymp k^{-3-2\alpha}$$

Lemma 5.5. *When $\alpha \neq 0$ we have that*

$$\widehat{w}_\alpha(k) = \Gamma(-(1+\alpha))^{-1} k^{-(1+\alpha)} (1 + \mathcal{O}k^{-1})$$

Combining these two lemmata we obtain the asymptotic representation for the Fourier coefficients of $w_\alpha r_\alpha$ namely

Lemma 5.6. *We have*

$$\widehat{w}_\alpha * \widehat{r}_\alpha(k) = \widehat{w}_\alpha(k)r_\alpha(0) + \mathcal{O}k^{-2-\alpha}$$

6. UNIFORM LAW OF LARGE NUMBERS AND ESTIMATES

We want to show the uniform strong law of large numbers (uniform SLLN) that we will use to obtain estimate for the posteriori distribution.

Theorem 6.1 (uniform SLLN). *Let I denote the interval $(\widehat{H} - 1/2, 1) \cap (0, 1)$. Then*

$$\mathbb{P}(\rho_n \rightarrow \rho_\infty \text{ uniformly on compact subsets of } I) = 1$$

where $\rho_n = \alpha \mapsto Q_n(\xi_n, \alpha)$ and $\rho_\infty = \alpha \mapsto F(\widehat{H}, \alpha)$.

This follows from the pointwise strong law of large numbers (Theorem 4.1). The extension to uniform convergence is done with the help of bounded variation with respect of the parameter α . This in effect can be reduced to monotonicity for auxiliary functions. Therefore, we consider how the family of derivatives $(\rho'_n)_n$ behaves.

Lemma 6.2. *For every α we have*

$$\partial_\alpha T_n(g_\alpha)^{-1} = -T_n(g_\alpha)^{-1}(\partial_\alpha T_n(g_\alpha))T_n(g_\alpha)^{-1}.$$

We first consider the part when $\alpha > 0$ (in our case $\alpha = 0$ corresponds to identity matrices so the case is trivial).

Lemma 6.3. *There exists an $\lambda_1 > 0$ and an $\lambda_2 > 0$ such that for every n the following estimates hold:*

- (1) $\forall \alpha \in (0, \frac{1}{2}): T_n(g_\alpha)^{-1} \leq \lambda_1 I_n$
- (2) $\forall \alpha \in (0, \frac{1}{2}): \partial_\alpha T_n(g_\alpha) \geq -\lambda_2 I_n$,
- (3) $\forall \alpha \in (0, \frac{1}{2}): \partial_\alpha T_n(g_\alpha)^{-1} \leq \lambda_1^2 \lambda_2 I_n$

The part when $\alpha < 0$ is similar, but in this case the zero in the symbol causes the sequence $(T_n(g_\alpha)^{-1})_n$ become unbounded and to handle that we need to estimate the matrices with unbounded symbols.

Lemma 6.4. *For every $\gamma \in (-\frac{1}{2}, 0)$ there exists a $\lambda_3 > 0$ and a $\lambda_4 > 0$ such that for every n the following estimates hold:*

- (1) $\forall \alpha \in [\gamma, 0): T_n(g_\alpha)^{-1} \leq \lambda_3 T_n(g_\gamma)^{-1}$
- (2) $\forall \alpha \in (-\frac{1}{2}, 0): \partial_\alpha T_n(g_\alpha) \geq -\lambda_4 T_n(g_\alpha)$
- (3) $\forall \alpha \in [\gamma, 0): \partial_\alpha T_n(g_\alpha)^{-1} \leq \lambda_3 \lambda_4 T_n(g_\gamma)^{-1}$.

With these estimates we can show the equicontinuity of the family $\{\rho_n\}$. First we construct auxiliary increasing family of functions.

Lemma 6.5. *Suppose that for some fixed sequence (z_n) , for some γ as in Lemma 6.4 and for some $\alpha_+ \in (\gamma, 1/2)$, the function*

$$m(\alpha) := \sup_n Q_n(z_n, \alpha)$$

is finite for $\alpha = \gamma$ and $\alpha = \alpha_+$. Suppose further that $0 < c \leq \|z_n\| \leq C$ for all n . Let $M = (\lambda_1^2 \lambda_2) \vee (\lambda_3 \lambda_4 m(\gamma)/c^2)$. Then auxiliary functions $\tilde{Q}_n(\alpha) := m(\gamma) - Q_n(z_n, \alpha) + M\alpha \|z_n\|^2$ have the following properties:

- (1) for every n the function \tilde{Q}_n is increasing and continuous on $[\gamma, \alpha_+]$.
- (2) the functions \tilde{Q}_n are uniformly bounded from below with lower bound 0
- (3) the functions \tilde{Q}_n are uniformly bounded from above with upper bound $m(\alpha_+) + M\alpha_+ C^2$.

Proof. This follows immediately from Lemmata 6.3 and 6.4 since for $\alpha > 0$ we know

$$\partial_\alpha \tilde{Q}_n(\alpha) \geq -\lambda_1^2 \lambda_2 \|z_n\|^2 + M \|z_n\|^2 \geq 0$$

and when $\alpha \in [\gamma, 0)$ we have

$$\partial_\alpha \tilde{Q}_n(\alpha) \geq -\lambda_3 \lambda_4 m(\gamma) + M \|z_n\|^2 \geq c^2(-\lambda_3 \lambda_4 m(\gamma)/c^2 + M) \geq 0.$$

□

We can now prove the uniform convergence for the auxiliary functions by Helly's Selection Theorem.

Lemma 6.6. *Suppose for fixed $z = (z_n)$ we know that on a dense subset J of the interval $[\gamma, \alpha_+]$*

$$\forall \alpha \in J: \lim_{n \rightarrow \infty} \tilde{\rho}_n(\alpha) = \rho_\infty(\alpha)$$

where $\tilde{\rho}_n(\alpha) = Q_n(z_n, \alpha)$. If in addition $\{\alpha_+, 0, \gamma\} \subset J$, then the sequence $(\tilde{\rho}_n)$ converges to ρ_∞ uniformly on $[\gamma, \alpha_+]$.

Proof. Since $\rho_n(0) = \|z_n\|^2$ converges to $\rho_\infty(0) \in (0, \infty)$, we can without a loss of generality assume that $0 < c \leq \|z_n\| \leq C < \infty$ for all n since the condition could be violated only finitely many times and we could replace $\tilde{\rho}_n$ by $\tilde{\rho}_{n+N}$.

Since γ and α_+ are in J we may without a loss of generality assume that every $f \in \{\tilde{\rho}_n\} \cup \{\rho_\infty\}$ the function f is increasing and uniformly bounded from above and from below. This follows by considering functions $f + \kappa$ instead, where $\kappa(\alpha) = M\alpha$.

The functions $\tilde{\rho}_n + \kappa$ are increasing and uniformly bounded from above and below by Lemma 6.5 for large enough $M > 0$. Moreover, by taking the M even larger, if necessary, we can assume the same for the function $\rho_\infty + \kappa$. Furthermore, it is enough to show the uniform convergence for these functions.

So let us suppose that all the functions are increasing, continuous and uniformly bounded from above and below. Choose any subsequence $(\phi_n) \subset (\tilde{\rho}_n)$. By Helly's Selection Theorem the sequence (ϕ_n) has a subsequence $(\tilde{\phi}_n)$ that converge pointwise to an increasing function ϕ_∞ on $[\gamma, \alpha_+]$. Moreover, the convergence is uniform if the function ϕ_∞ is

continuous. Since $\tilde{\phi}_n$ converges to ρ_∞ on J , we see that $\phi_\infty(\alpha) = \rho_\infty(\alpha)$ for every $\alpha \in J$. For every point of continuity β of ϕ_∞ , we know that

$$\phi_\infty(\beta) = \sup_{\alpha < \beta, \alpha \in J} \rho_\infty(\alpha) = \rho_\infty(\beta).$$

If β would be a point of discontinuity, we would know that

$$\sup_{\alpha < \beta, \alpha \in J} \phi_\infty(\alpha) < \inf_{\alpha > \beta, \alpha \in J} \phi_\infty(\alpha)$$

but since both left and right hand sides are $\rho_\infty(\beta)$ by continuity of ρ_∞ , we have a contradiction. Therefore, we may deduce that $\phi_\infty = \rho_\infty$ and thus $(\tilde{\phi}_n)$ converges uniformly to ρ_∞ on $[\gamma, \alpha_+]$. This in turn implies that $(\tilde{\rho}_n)$ converges uniformly to ρ_∞ on $[\gamma, \alpha_+]$ since the uniform convergence is topological convergence. \square

With these lemmata, we obtain the Theorem 6.1 in a straight forward manner.

Proof of Theorem 6.1. Let $I' = I - 1/2 = (\beta_-, \beta_+)$ where I is as stated in the claim. Let $k > 0$ and choose $\alpha_{-,k}, \alpha_{+,k} \in \mathbb{Q}$ such that $\beta_- < \alpha_{-,k} < \beta_- + \frac{1}{k} < \beta_+ - \frac{1}{k} < \alpha_{+,k} < \beta_+$.

Choose a countable dense set $J = [\alpha_{-,k}, \alpha_{+,k}] \cap \mathbb{Q}$. Theorem 4.1 implies that

$$\tilde{\Omega} = \{\forall \alpha \in J: \lim_n \rho_n(\alpha) = \rho_\infty(\alpha)\}$$

is an almost sure event. Let $z_n = \xi_n(\omega)$ for $\omega \in \tilde{\Omega}$. Application of Lemma 6.6 implies that $\tilde{\rho}_n := \rho_n(\omega)$ converges uniformly to ρ_∞ on $I' \cap [\gamma, \alpha_+]$. Therefore, we deduce that

$$\tilde{\Omega} \subset \Omega_k := \{\rho_n \rightarrow \rho_\infty \text{ uniformly on } [\alpha_{-,k}, \alpha_{+,k}]\}$$

This implies that $\mathbb{P}(\bigcap_k \Omega_k) = 1$ and the claim follows. \square

Since the convergence takes place not in the whole interval $(-1/2, 1/2)$ we need some estimates to handle the remaining parts so that we can at least obtain the parameter estimation result. For this we would need to obtain an upper bound for the quadratic forms $\rho_n(\alpha)$ for all α . When $\alpha > 0$ we have an upper bound by Lemma 6.3, but for $\alpha < 0$ the zero in the symbol causes the singularity that caused an restriction.

Lemma 6.7. *For every $\varepsilon > 0$ there exists a constant $\lambda_5 > 0$ such that every $\alpha \in [-\frac{1}{2} + \varepsilon, -\varepsilon]$ there exists a symbol \tilde{g}_α*

- (1) $\tilde{g}_\alpha \leq g_\alpha \leq \lambda_5 \tilde{g}_\alpha$ and
- (2) $\partial_\alpha \tilde{g}_\alpha \geq 0$
- (3) $\lambda_5^{-1} T_n(\tilde{g}_\alpha)^{-1} \leq T_n(g_\alpha)^{-1} \leq T_n(\tilde{g}_\alpha)^{-1}$

The existence of these auxiliary symbols \tilde{g}_α imply that $T_n(g_\alpha)^{-1} \geq \lambda_5^{-1} T_n(\tilde{g}_\alpha)^{-1}$ for every $\alpha < \beta < 0$ and therefore, we obtain

Theorem 6.8. *Let I , ρ_n and ρ_∞ be as in Theorem 6.1. Then*

$$\mathbb{P}(\rho_n \rightarrow \rho_\infty \text{ uniformly on compact subsets of } I \text{ and } \liminf_{n \rightarrow \infty} \rho_n(t) = \infty) = 1.$$

7. PARAMETER ESTIMATION FROM THE POSTERIOR

According to Theorem 6.1 there exists an almost sure event $\Omega' \subset \Omega$ such that $\forall \omega \in \Omega', \forall \hat{H} \in (0, 1), \forall \alpha \in (0, 1)$ if $\alpha > \hat{H} - \frac{1}{2}$ then

$$(24) \quad \lim_{n \rightarrow \infty} \frac{n^{2(\alpha - \hat{H})} Q_n(z_n, \alpha)}{n^{2(\alpha - \hat{H}) + 1} F(\alpha, \hat{H})} = 1.$$

where $z_n := \xi_n(\hat{H})(\omega)$ and $Q_n(z_n, \alpha) := \langle z_n, T_n(f_\alpha)^{-1} z_n \rangle$. The deterministic function F is the expected value

$$F(\alpha, \beta) := \int_{\mathbb{T}} \frac{f_\beta(t)}{f_\alpha(t)} dt.$$

Proposition 7.1. *We have $\forall \alpha \in (0, 1), \varepsilon > 0$ the determinant has the asymptotic estimate*

$$(25) \quad \left| \frac{|T_n(f_\alpha)|}{G(\alpha)^n (1+n)^{(1-2\alpha)^2/4}} - E(H) \right| \leq \frac{C(\alpha)}{(1+n)^{1-\varepsilon}},$$

for large enough n where $C, G, E: (0, 1) \rightarrow \mathbb{R}^+ \setminus \{0\}$ are continuous functions with respect to H .

Proof. The claim follows directly from [14, Theorem 2.5], since in our case the symbol f_H has exactly one Fisher–Hartwig singularity and by Lemma 5.4, the assumptions of [14, Theorem 2.5] are fulfilled with parameters $\delta = \gamma = \alpha$. This implies the claim. \square

Without a loss of generality, we may assume that the set \hat{B} is an interval $[0, \beta)$ for some $\beta < 1$. According to Proposition 7.1 we have $\forall \alpha \in (0, 1)$ that

$$(26) \quad \log |T_n(f_\alpha)| = \mathcal{O}n$$

Therefore, by Dominated Convergence

$$(27) \quad \begin{aligned} & \lim_{n \rightarrow \infty} C_n(z_n) \int_{\alpha_-}^{\beta} n^{n\alpha} |T_n(f_\alpha)|^{-1/2} \exp\left(-\frac{1}{2} n^{2(\alpha - \hat{H})} Q_n(z_n, \alpha)\right) d\alpha \\ &= \lim_{n \rightarrow \infty} C_n(z_n) \int_{\alpha_-}^{\beta} \exp\left(\alpha n \ln n - \frac{1}{2} n^{2(\alpha - \hat{H}) + 1} F(\alpha, \hat{H})\right) d\alpha, \end{aligned}$$

where $\alpha_- = (\hat{H} - 1/2)_+$. Similarly, we can estimate

$$(28) \quad \begin{aligned} & \limsup_{n \rightarrow \infty} C_n(z_n) \int_0^{\alpha_-} n^{n\alpha} |T_n(f_\alpha)|^{-1/2} \exp\left(-\frac{1}{2} n^{2(\alpha - \hat{H})} Q_n(z_n, \alpha)\right) d\alpha \\ & \leq \limsup_{n \rightarrow \infty} C_n(z_n) \int_0^{\alpha_-} \exp(\alpha n \ln n) d\alpha. \end{aligned}$$

We know that the function $\alpha \mapsto F(\alpha, \widehat{H})$ is continuous and $F(\alpha, \alpha) = 1$. If we replace the function $\widehat{F} := \alpha \mapsto F(\alpha, \widehat{H})$ with a constant function $\mathbf{1} := \alpha \mapsto 1$ the function

$$K_n(L)(\alpha) := \alpha n \log n - \frac{1}{2} n^{2(\alpha - \widehat{H}) + 1} L(\alpha)$$

inside the exponent function on the right-hand side of the equation (27) would become

$$k_n(\alpha) := K_n(\mathbf{1})(\alpha) = \alpha n \log n - \frac{1}{2} n^{2(\alpha - \widehat{H}) + 1}.$$

Differentiation with respect to α reveals that

$$k'_n(\alpha) = n \log n (1 - n^{2(\alpha - \widehat{H})})$$

which is negative when $\alpha > \widehat{H}$ and positive when $\alpha < \widehat{H}$. It has a unique zero at $\alpha = \widehat{H}$ which means that the function k_n has a unique global maximum at $\alpha = \widehat{H}$.

Since $k_n = K_n(\mathbf{1})$ has a unique global maximum at $\alpha = \widehat{H}$ and the functions \widehat{F} and \widehat{F}' are continuous, we could expect that the function $K_t(\widehat{F})$ would also have a global maximum *near* the point $\alpha = \widehat{H}$. In order to show that this is, indeed, the case we differentiate $\kappa_n := K_n(\widehat{F})$ which gives that

Lemma 7.2. *We have*

$$\kappa'_n(\alpha) = n \log n (1 - n^{2(\alpha - \widehat{H})}) \varphi_n(\alpha)$$

and

$$\kappa''_n(\alpha) = -n(\log n)^2 n^{2(\alpha - \widehat{H})} \psi_n(\alpha)$$

where

$$\varphi_n(\alpha) := \widehat{F}(\alpha) + \frac{\widehat{F}'(\alpha)}{2 \log n}$$

and

$$\psi_n(\alpha) := 2\varphi_n(\alpha) + \frac{\widehat{F}'(\alpha)}{\log n} + \frac{\widehat{F}''(\alpha)}{2(\log n)^2}$$

In the sequel, we use maximum and minimum operators defined as

$$M(f) := \sup \{ f(x) \mid x \in (\widehat{H} - 1/2 + \varepsilon, 1) \}$$

and

$$m(f) := \inf \{ f(x) \mid x \in (\widehat{H} - 1/2 + \varepsilon, 1) \}.$$

We have to cut out a small neighbourhood of $\widehat{H} - 1/2$ since the function \widehat{F} explodes at $\widehat{H} - 1/2$.

Lemma 7.3. *There is $N \in \mathbb{N}$ such that $\forall n \geq N$ we have*

$$\frac{1}{2} m(\widehat{F}) \leq \varphi_n \leq 2M(\widehat{F}) \quad \text{and} \quad \psi_n \geq \frac{1}{2} m(\widehat{F}).$$

Lemma 7.4. *The function κ_n is concave for every n large enough.*

Proof. This follows from Lemmata 7.2 and 7.3. \square

Lemma 7.5. *Let $\alpha_+(n) := \widehat{H} + (\log n)^{-1}(\log 2 - \log m(\widehat{F}))$ and let $\alpha_-(n) := \widehat{H} - (\log n)^{-1}(\log 2 + \log M(\widehat{F}))$. When N is defined as in Lemma 7.3 then $\forall n \geq N$ the equation $\kappa'_n(\alpha) = 0$ has a unique solution α_n inside the interval $[\alpha_-(n), \alpha_+(n)]$.*

Proof. Suppose $\alpha > \alpha_+(n)$. Then by Lemma 7.3 we have

$$\kappa'_n(\alpha) < n \log n \left(1 - \frac{1}{2}m(\widehat{F})n^{2(\alpha_+(n)-\widehat{H})}\right).$$

Since

$$n^{2(\alpha_+(n)-\widehat{H})} = \exp(\log(2/m(\widehat{F}))) = 2/m(\widehat{F})$$

we see that $\kappa'_n(\alpha) < 0$. In the same way, if we suppose $\alpha < \alpha_-(n)$, we similarly

$$\kappa'_n(\alpha) > n \log n \left(1 - 2M(\widehat{F})n^{2(\alpha_-(n)-\widehat{H})}\right) \geq 0.$$

Therefore, for every $n \geq N$ the continuous and decreasing function κ'_n changes sign on interval $[\alpha_-(n), \alpha_+(n)]$. This gives the claim. \square

We can asymptotically solve the equation $\kappa'_n(\alpha_n) = 0$.

Lemma 7.6. *We have that*

$$\alpha_n = \widehat{H} - \frac{\widehat{F}'(\widehat{H})}{4(\log n)^2} + \mathcal{O}(\log n)^{-3}$$

Remark. The numeric computations and more qualitative arguments indicate that $\widehat{F}'(\widehat{H}) > c > 0$ for every \widehat{H} . Therefore, the maximum aposteriori estimate is biased to left of the true value. Furthermore, we see that as n grows to infinity the maximum aposteriori estimate becomes asymptotically unbiased.

Since we need the values of the higher derivatives of κ_n at α_n , let's compute them.

Lemma 7.7. *We have that*

$$\kappa''_n(\alpha_n) = -2n(\log n)^2 c_n$$

and

$$\kappa^{(3)}_n(\alpha_n) = -4n(\log n)^2 s_n$$

where $c_n = 1 + \mathcal{O}(\log n)^{-1}$ and $s_n = 1 + \mathcal{O}(\log n)^{-1}$. Furthermore, we can estimate that

$$\forall \alpha \in U_n(\alpha_n): |\kappa^{(4)}_n(\alpha)| \lesssim n(\log n)^4$$

where $U_n(\alpha_n) = \{|\alpha - \alpha_n| \ll (\log n)^{-1}\}$.

When $\alpha < \widehat{H} - \frac{1}{2} + \varepsilon$ we cannot use the derivatives to study the extremal points but we have a simple estimate for the function κ itself. Since \widehat{F} is positive function, for every $\alpha \in (\widehat{H} - \frac{1}{2}, \widehat{H} - \frac{1}{2} + \varepsilon)$ we have that

$$\kappa_n(\alpha) \leq \alpha n \log n \leq (\widehat{H} - \frac{1}{2} + \varepsilon) t \log t.$$

The same estimate holds on the interval $[0, \widehat{H} - \frac{1}{2}]$ as well. Since

$$\kappa_n(\alpha_n) \geq \kappa_n(\widehat{H}) = (\widehat{H} \log n - \frac{1}{2})n > (\widehat{H} - \frac{1}{2} + \varepsilon)n \log n$$

that holds when $\varepsilon < \frac{1}{4}$ and $\log n > 2$, we infer that κ_n has its global maximum at α_n .

This leads to the Laplace method type argument, since we rescale the maximum to be one. In other words, we define

$$I_n(\widehat{V}) := e^{-K_n(\widehat{F})(\alpha(n))} \int_{\widehat{V}} e^{K_n(\widehat{F})(\alpha)} d\alpha$$

and for the upper bound of the remainder part

$$J(n) := \left[\widehat{H} > \frac{1}{2} \right] e^{-K_n(\widehat{F})(\alpha(n))} \int_0^{\widehat{H} - \frac{1}{2}} e^{\alpha n \log n} d\alpha$$

Following the usual procedure, we divide the integration interval into tail parts and the main part. In the remaining part of this section we will denote the the left tail interval as $(0, \beta_-(n))$ and the right tail interval $(\beta_+(n), 1)$. The main part is the interval between $\beta_-(n)$ and $\beta_+(n)$.

The next lemma shows that we can express the upper bounds $\rho_{\pm}(n)$ of the tail errors in terms of the derivatives of κ_n .

Lemma 7.8. *For fixed n we have for every $\beta_-(n) < \alpha(n) < \beta_+(n)$ that*

$$I_n((0, \beta_-(n))) = \mathcal{O}\rho_-(n)$$

and

$$I_n((\beta_+(n), 1)) = \mathcal{O}\rho_+(n)$$

where $\rho_-(n) := 1/\kappa'_n(\beta_-(n))$ and $\rho_+(n) := -1/\kappa'_n(\beta_+(n))$.

The next lemma shows that we can express the upper bounds $\rho_{\pm}(n)$ of the tail errors in terms of the distance of $\beta_{\pm}(n)$ from the zero point α_n .

Lemma 7.9. *When the distance of $\beta_{\pm}(n)$ from α_n is $\varepsilon_n n^{-1/2}(\log n)^{-1}$ with $\varepsilon_n \ll n^{1/2}$ the upper bounds $\rho_{\pm}(n)$ of tail estimates are of order*

$$\rho_{\pm}(n) \asymp \frac{1}{\varepsilon_n n^{1/2} \log n}$$

The only remaining interval is $V = [\beta_-(n), \beta)$ for some $\beta < \beta_+(n)$. The function

$$k_n(\alpha) := K_n(\widehat{F})(\alpha + \alpha(n)) - K_n(\widehat{F})(\alpha(n))$$

has a zero at $\alpha = 0$. Since $\alpha(n)$ is the global maximum, we know that $k'_n(0) = 0$ and $k''_n(0) = -nc_n(\log n)^2 < 0$. This leads to the following result.

Lemma 7.10. *Suppose $|\beta_{\pm} - \alpha_n| = \varepsilon_n n^{-1/2}(\log n)^{-1}$ for some $\varepsilon_n \ll n^{1/2}$. Let τ be a change of variable $\tau_n(\beta) = (\beta - \alpha_n) \log n \sqrt{nc_n}$ and let $\varepsilon = n^{-1/2} \lambda_n^{-1}$ for some $\lambda_n = o(1)$. Then*

$$\frac{1}{\sqrt{2\pi}} I_n([\beta_-(n), \beta)) = \frac{1}{\log n \sqrt{c_n n}} \left(\Phi \circ \tau(\beta) - \lambda_-(n) \right) + \mathcal{O} \frac{\varepsilon_n}{n \log n}.$$

where $\lambda_-(n) = \Phi \circ \tau(\beta_-(n))$.

Lemma 7.11. *The optimal choice for the ε_n in order to minimize the error estimate is $\varepsilon_n \asymp n^{1/4}$.*

Proof. According to Lemma 7.10 the error term is increasing in ε_n . The errors coming from the tails are decreasing in ε_n according to Lemma 7.9. Either we get the minimum at the end points or at the point of crossing.

When ε_n is nearly 1 then the error from the main part is almost of order $n^{-1}(\log n)^{-1}$ and the tail error is almost of order $n^{-1/2}(\log n)^{-1}$.

When ε_n is just under the upper bound $n^{1/2}$ the error from tails is almost than $n^{-1}(\log n)^{-1}$ but the error from the main part is only of order $n^{-1/2}(\log n)^{-1}$.

Therefore, the minimum is obtained at the point of crossing. This happens when

$$\frac{\varepsilon_n}{n \log n} = \frac{1}{\varepsilon_n n^{1/2} \log n}$$

and the claim follows. \square

With these choices we have found that

Lemma 7.12. *We have*

$$\mathbf{P}(H \leq t \mid \mathbf{Y}_n(\omega)) = \Phi(t_n)(1 + \mathcal{O}n^{-1/4})$$

where $t_n = (t - \alpha_n) \sqrt{c_n n} \log n$.

Moreover, this implies that for every $\widehat{B} = (0, t)$ we have that

$$\mathbf{P}(H \leq t \mid \mathbf{Y}_n(\omega)) = \Phi(t_n)(1 + \mathcal{O}n^{-1/4})$$

where $t_n = (t - \alpha_n) \sqrt{c_n n} \log n$. In order to make both sides coincide better we denote $\widetilde{H}_n := (H - \alpha_n) \sqrt{c_n n} \log n$. Then

$$\{H \leq t\} = \{H - \alpha_n \leq t - \alpha_n\} = \{\widetilde{H}_n \leq t_n\}.$$

As the final conclusion we get

Theorem 7.13. *There exists an $\alpha_n \in (0, 1)$, a bounded sequence c_n and $M > 0$ such that $|\alpha_n - \hat{H}| \leq M/(\log n)^2$. Furthermore, there exists an almost sure event $\Omega' \subset \Omega$ so that for every $\omega \in \Omega'$ the conditional distribution of*

$$\mathbf{P}(\tilde{H}_n \leq t \mid \mathbf{Y}_n(\omega))$$

is asymptotically standard normal distribution $\Phi(t)$ when

$$\tilde{H}_n := (H - \alpha_n)\sqrt{c_n n \log n}.$$

Corollary 7.14. *The conditional mean and the maximum a posteriori estimates of H are both equal to α_n asymptotically. Moreover, both are asymptotically unbiased estimators of \hat{H} . The conditional variance has a formula*

$$\mathbf{E}((\hat{H} - \alpha_n)^2 \mid \mathbf{Y}_n) = \frac{1}{c_1(n)n(\log n)^2}(1 + \mathcal{O}n^{-1/4}).$$

Remark. Since the posteriori variance converges faster to zero than the expectation, we note that for large n the posteriori solution would falsely give confidence intervals that will not intersect with the true value. However, first few digits would be still reliable.

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APPENDIX A. PROOFS OF AUXILIARY RESULTS IN SECTION 3

In this section we prove the technical results that were mentioned in Section 3. For the proof of Lemma 3.3 we need a following result.

Lemma A.1. *We have for $N \geq 1$ and any matrix B that*

$$\Theta(B, N) = (B^s : C)\Theta(N - 1) + 2(N - 1)\Theta(B^s C A, N - 1).$$

where $B^s = \frac{1}{2}(B + B^\top)$.

With the help of this auxiliary result we can prove Lemma 3.3.

Proof of Lemma 3.3. We will show that

$$(29) \quad \Theta(N) = \sum_{j=1}^k \beta_j + 2^k(N - 1)^k \Theta((AC)^k A, N - k)$$

for every $k = 1, \dots, N - 1$ where

$$\beta_j := 2^{j-1}(N-1)^{\underline{j-1}} R_j \Theta(N-j).$$

The proof is by induction with respect to k . When $k = 1$, this is the special case of Lemma A.1 for $B = A$.

Assuming that identity (29) holds for $k < N - 1$, then

$$\Theta(N) = \sum_{j=1}^k \beta_j + 2^k(N-1)^{\underline{k}} \Theta((AC)^k A, N-k).$$

Since by Lemma A.1

$$\begin{aligned} \Theta((AC)^k A, N-k) &= R_{k+1} \Theta(N-(k+1)) \\ &\quad + 2(N-(k+1)) \Theta((AC)^{k+1} A, N-(k+1)) \end{aligned}$$

the identity (29) holds for $k+1 \leq N-1$, as well, and therefore, by induction for every $k = 1, \dots, N-1$. When $k = N-1$, the last term on the right-hand side of the identity (29) reduces to

$$2^{N-1}(N-1)^{\underline{(N-1)}} \Theta((AC)^{N-1} A, 1) = \beta_N$$

and the claim follows. \square

Now that we know that Lemma A.1 is useful, we can now continue proving it. The proof relies on Isserlis–Wick Theorem.

Proof of Lemma A.1. First we note that $\Theta(B, N) = \Theta(B^s, N)$ so we may assume that B is symmetric.

We recall and extend some notations from the introduction. We call multi-indices as *words*. For every word $\rho = (\rho_j) \in \mathbb{N}^k$ of length k and every subset $J \subset \{1, 2, \dots, k\}$ of cardinality m the word $\bar{\rho}_J$ is the word where the letters ρ_j where $j \in J$ are removed. Furthermore, for every permutation σ on J we will denote ρ_σ the word consisting the letters ρ_j where $j \in J$ in the order given by the permutation σ .

Using these notations with the definition of the $\Theta(B, N)$ we can write

$$\Theta(B, N) = \sum_{\rho} B_{\rho_{(1,2)}} A_{\bar{\rho}_{\{1,2\}}}^{\otimes(N-1)} \mathbf{E} \xi_{\rho}^{\otimes 2N}.$$

By the Isserlis–Wick Theorem, the expectation can be written as a sum

$$\mathbf{E} \xi_{\rho}^{\otimes 2N} = \sum_{k=2}^{2N} C_{\rho_{(1,k)}} \mathbf{E} \xi_{\bar{\rho}_{\{1,k\}}}^{\otimes 2(N-1)}$$

The term when $k = 2$ gives

$$\sum_{\rho} B_{\rho_{(1,2)}} A_{\bar{\rho}_{\{1,2\}}}^{\otimes(N-1)} \mathbf{E} \xi_{\bar{\rho}_{\{1,2\}}}^{\otimes 2(N-1)} = (B : C) \Theta(N-1)$$

which is the first term in the claim. The remaining terms can be written as

$$2 \sum_{k=2}^N \sum_{\rho} B_{\rho(1,2)} A_{\rho(2k-1,2k)} A_{\bar{\rho}\{1,2,2k-1,2k\}}^{\otimes(N-2)} C_{\rho(1,2k-1)} \mathbf{E}_{\xi_{\bar{\rho}\{1,2k-1\}}}^{\otimes 2(N-1)}$$

by using the change of variables $\rho_{2k-1} \leftrightarrow \rho_{2k}$ and the symmetry of A . The change of variables $\rho_3 \leftrightarrow \rho_{2k-1}$ and $\rho_4 \leftrightarrow \rho_{2k}$ show that the previous sum reduces to

$$\begin{aligned} & 2(N-1) \sum_{\rho} B_{\rho(1,2)} A_{\rho(3,4)} A_{\bar{\rho}\{1,2,3,4\}}^{\otimes(N-2)} C_{\rho(1,3)} \mathbf{E}_{\xi_{\bar{\rho}\{1,3\}}}^{\otimes 2(N-1)} \\ &= 2(N-1) \sum_{\sigma} \sum_{\rho_1, \rho_3} B_{\rho_1 \sigma_1} A_{\rho_3 \sigma_2} A_{\bar{\sigma}\{1,2\}}^{\otimes(N-2)} C_{\rho_1 \rho_3} \mathbf{E}_{\xi_{\sigma}}^{\otimes 2(N-1)} \\ &= 2(N-1) \sum_{\sigma} B C A_{\sigma(1,2)} A_{\bar{\sigma}\{1,2\}}^{\otimes(N-2)} \mathbf{E}_{\xi_{\sigma}}^{\otimes 2(N-1)} \end{aligned}$$

where we used the symmetricity of B . Since the last line coincides with

$$2(N-1) \Theta(BCA, N-1)$$

the claim follows. \square

The proof of recursion equation (Lemma 3.4) is straightforward induction argument.

Proof of Lemma 3.4. The result follows by induction. When $k = 1$, the result follows immediately from the definition (8) and the Lemma 3.3.

Let us assume that the claim holds for $k < n - 1$. Since

$$\sum_{j=k}^n a_k(j, n) \Theta(n-j) = a_k(k, n) \Theta(n-k) + \sum_{j=k+1}^n a_k(j, n) \Theta(n-j)$$

we can expand the first term on the right-hand side with Lemma 3.3 and by change of summation variable $j' = j + k$ we obtain

$$\Theta(n) = \sum_{j=k+1}^n a_k(k, n) a_1(j-k, n-k) + \sum_{j=k+1}^n a_k(j, n) \Theta(n-j)$$

and the claim follows. \square

Next we will solve the recursion equation for a and therefore, the Θ . This is the content of Lemma 3.5. We need some auxiliary functions to solve the recursion easily. First we denote

$$(30) \quad I(n, m) = \bigcup_{j=1}^m J(n, j).$$

With this we can define for $j \leq n$ and $\mathbf{k} \in I(m, j)$

$$(31) \quad \Xi(\mathbf{k}, j, n) = a_1(j - s_m(\mathbf{k}), n - s_m(\mathbf{k})) \prod_{i=1}^m a_1(\mathbf{k}_i, n - s_{i-1}(\mathbf{k}))$$

Lemma A.2. *We have for $1 \leq k \leq j \leq n$ that*

$$(32) \quad a_k(j, n) = \sum_{m=0}^{k-1} \sum_{\mathbf{k}} \Xi(\mathbf{k}, j, n) [\mathbf{k} \in I(m, k-1)].$$

In particular,

$$\Theta(n) = \sum_{m=0}^{n-1} \sum_{\mathbf{k}} \Xi(\mathbf{k}, n, n) [\mathbf{k} \in I(m, n-1)].$$

Since we have an explicit formula for $\Xi(\mathbf{k}, n, n)$ and for a_1 in terms of $R^{\mathbf{k}}$, we obtain a more explicit formula for Θ .

Proof of Lemma A.2. We show the identity (32) by induction with respect to k . When $k = 1$, the identity follows from the fact that $I(0, 0) = \{\emptyset\}$ consists of a single element, namely the empty word \emptyset . Since $s_0(\emptyset) = 0$ and the empty product is 1, we notice that the right-hand side of (32) reduces to

$$\Xi(\emptyset, j, n) = a_1(j - s_0(\emptyset), n - s_0(\emptyset)) \prod_{\emptyset} \cdots = a_1(j, n)$$

and the claim holds for $k = 1$.

Let us now suppose that the claim holds for $k < n$. By Lemma 3.4 we have

$$(33) \quad \begin{aligned} a_{k+1}(j, n) &= a_k(k, n) a_1(j - k, n - k) + a_k(j, n) \\ &= \sum_{m=0}^{k-1} \sum_{\mathbf{k} \in I(m, k-1)} \Xi(\mathbf{k}, j, n) + \Xi(\mathbf{k}, k, n) a_1(j - k, n - k) \end{aligned}$$

where the last identity follows by the induction assumption. When $\mathbf{k} \in I(m, k-1)$ we define a new word $\bar{\mathbf{k}}$ by adding a single letter in the end

$$\bar{\mathbf{k}} = \mathbf{k} \& \{k - s_m(\mathbf{k})\}.$$

We notice that the mapping $\mathbf{k} \mapsto \bar{\mathbf{k}}$ defines a bijection from $I(m, k-1)$ onto $J(m+1, k)$. Furthermore, we notice that for $\mathbf{k} \in I(m, k-1)$ we have

$$\begin{aligned} \Xi(\mathbf{k}, k, n) a_1(j - k, n - k) &= a_1(j - k, n - k) \prod_{i=1}^{m+1} a_1(\bar{\mathbf{k}}_i, n - s_{i-1}(\bar{\mathbf{k}})) \\ &= \Xi(\bar{\mathbf{k}}, j, n). \end{aligned}$$

Therefore, since $\mathbf{k} \mapsto \bar{\mathbf{k}}$ is a bijection, we obtain

$$(34) \quad \sum_{m=0}^{k-1} \sum_{\mathbf{k} \in I(m, k-1)} \Xi(\mathbf{k}, k, n) a_1(j - k, n - k) = \sum_{m=1}^k \sum_{\mathbf{k} \in J(m, k)} \Xi(\mathbf{k}, j, n)$$

Moreover, since $I(0, k) = \{\emptyset\}$ and $I(k, k) = J(k, k)$ and

$$[\mathbf{k} \in I(m, k-1)] + [\mathbf{k} \in J(m, k)] = [\mathbf{k} \in I(m, k)]$$

the induction hypothesis follows by combining identities (33) and (34). This proves the claim. \square

We can apply the previous lemma (Lemma A.2) to solve the recursion equation (Equation (8)) for Θ .

Proof of Lemma 3.5. By Lemma A.2 we have

$$\Theta(n) = \sum_{m=0}^{n-1} \sum_{\mathbf{k}} \Xi(\mathbf{k}, n, n) [\mathbf{k} \in I(m, n-1)].$$

We note that for $\mathbf{k} \in I(m, n-1)$ it holds that

$$\Xi(\mathbf{k}, n, n) = \prod_{i=1}^{m+1} a_1(\bar{\mathbf{k}}_i, n - s_{i-1}(\bar{\mathbf{k}}))$$

where $\bar{\mathbf{k}} = \mathbf{k} \& \{n - s_m(\mathbf{k})\}$ as in the proof of Lemma A.2. Since the mapping $\mathbf{k} \mapsto \bar{\mathbf{k}}$ is a bijection from $I(m, n-1)$ onto $J(m+1, n)$ the claim follows by using the facts that

$$a_1(j, n) = 2^{j-1}(n-1)^{j-1}R_j$$

and the definition of the functions s_j . \square

The representation of Lemma 3.5 can be used to obtain the representation of given by Lemma 3.6.

Proof of Lemma 3.6. The Lemma 3.5 immediately implies that

$$\Theta(n) = \sum_{m=1}^n \sum_{l=0}^m 2^{n-m} R_1^l \sum_{\mathbf{k} \in J_l(m, n)} \prod_{j=1}^m q(j, \mathbf{k}_j, \mathbf{k})$$

where

$$q(j, k, \mathbf{k}) = (R_k[k \neq 1] + [k = 1])(n - s_{j-1}(\mathbf{k}) - 1)^{k-1}.$$

Note that every $\mathbf{k} \in J_l(m, n)$ can be uniquely represented by giving the locations and values of the indices different from 1. In particular, there is a bijection $\pi = (\pi_j)$ from

$$J_0(m-l, n-l) \times L(m-l, m) \rightarrow J_l(m, n)$$

given by

$$\pi_k(\bar{\mathbf{k}}, \lambda) = 1 + \sum_j (\bar{\mathbf{k}}_j - 1) [\lambda_j = k].$$

Therefore,

$$\begin{aligned}
& \sum_{\mathbf{k} \in J_l(m, n)} \prod_{j=1}^m q(j, \mathbf{k}_j, \mathbf{k}) \\
&= \sum_{\bar{\mathbf{k}} \in J_0(m-l, n-l)} \sum_{\lambda \in L(m-l, m)} \prod_{1 \leq j \leq m-l} q(\lambda_j, \bar{\mathbf{k}}_j, \pi(\bar{\mathbf{k}}, \lambda)) \\
&= \sum_{\mathbf{k} \in J_0(m-l, n-l)} R^{\mathbf{k}} \Lambda(n, \mathbf{k})
\end{aligned}$$

and the claim follows. \square

In order to prove Lemma 3.7 we need some more auxiliary results. We first introduce the word length function

$$(35) \quad \psi(\mathbf{k}) = \text{"length of } \mathbf{k} \text{"}$$

Next, we denote

$$(36) \quad w(n, \lambda, \mathbf{k}) := \prod_{j=1}^{\psi(\mathbf{k})} (n - (\lambda_j - j) - s_{j-1}(\mathbf{k}) - 1)^{\underline{\mathbf{k}_j - 1}}$$

for every $\lambda \in L(\psi(\mathbf{k}), m)$.

The auxiliary function w can be written in a closed form with the help of induction. We will provide the large step reduction lemma, small step reduction and the base step in auxiliary lemmata.

Lemma A.3. *Let $M = m - l \geq 1$. For every $\mathbf{k} \in J_0(M, n - l)$ we have*

$$\sum_{\lambda \in L(M, m)} w(n, \lambda, \mathbf{k}) = \binom{n}{n-l} (n-l-1)! \prod_{2 \leq j \leq M} \frac{1}{s_M(\mathbf{k}) - s_{j-1}(\mathbf{k})}$$

In order to show this we need a reduction lemma that reduces the length of word \mathbf{k} .

Lemma A.4. *Let $M = m - l > 1$. For every $\mathbf{k} \in J_0(M, n - l)$ we have*

$$\sum_{\lambda \in L(M, m)} w(n, \lambda, \mathbf{k}) = \mathbf{k}_M^{-1} \sum_{\lambda \in L(M-1, m-1)} w(n, \lambda, \bar{\mathbf{k}})$$

where $\bar{\mathbf{k}} \in J_0(M-1, n-l)$ and is defined as

$$\bar{\mathbf{k}}_j := [j < M-1] \mathbf{k}_j + [j = M-1] (\mathbf{k}_M + \mathbf{k}_{M-1})$$

Furthermore, we need the base step lemma for one letter words.

Lemma A.5. *For every $\mathbf{k} \in J_0(1, n-l)$ we have*

$$\sum_{\lambda \in L(1, l+1)} w(n, \lambda, \mathbf{k}) = \binom{n}{\mathbf{k}_1} (\mathbf{k}_1 - 1)!$$

Now we can prove the representation lemma for the auxiliary function w .

Proof of Lemma A.3. When $M = 1$, we have $m = l + 1$ and $\mathbf{k}_1 = n - l$. Therefore, the claim follows from Lemma A.5.

Suppose the claim holds for $M = M_0 \geq 1$ and consider the case $M = M_0 + 1$. In this case, $m = M + l = M_0 + l + 1$. By Lemma A.4 we have

$$\begin{aligned} \sum_{\lambda \in L(M, m)} w(n, \lambda, \mathbf{k}) &= \mathbf{k}_M^{-1} \sum_{\lambda \in L(M_0, M_0 + l)} w(n, \lambda, \bar{\mathbf{k}}) \\ &= \mathbf{k}_M^{-1} \binom{n}{n-l} (n-l-1)! \prod_{2 \leq j \leq M_0} \frac{1}{s_{M_0}(\bar{\mathbf{k}}) - s_{j-1}(\bar{\mathbf{k}})} \end{aligned}$$

Since $s_{M_0}(\bar{\mathbf{k}}) = s_M(\mathbf{k})$ and $s_j(\bar{\mathbf{k}}) = s_j(\mathbf{k})$ for every $j < M_0$ and moreover, $\mathbf{k}_M = s_M(\mathbf{k}) - s_{M-1}(\mathbf{k})$ the induction claim follows and the claim is proved. \square

Next we prove the base step.

Proof of Lemma A.5. In this case, we have

$$w(n, \lambda, \mathbf{k}) = (n-j)^{\underline{k-1}}$$

where $j = \lambda_1 \in \{1, \dots, l+1\}$ and $k = \mathbf{k}_1$. Since $\mathbf{k} \in J_0(1, n-l)$ we have $k = n-l$. Therefore the sum in this case reduces to

$$\sum_{j=1}^{l+1} (n-j)^{\underline{k-1}} = \sum_{j=n-l-1}^{n-1} j^{\underline{k-1}} = \sum_{j=k}^{n-1} j^{\underline{k-1}} = (k-1)! \sum_{j=k}^{n-1} \binom{j}{k-1}$$

and since the last sum equals to $\binom{n}{k}$ the claim follows. \square

The reduction lemma (Lemma A.4) will be shown next.

Proof of Lemma A.4. Since n is fixed throughout the proof, we will drop it from the argument lists of functions.

We split the $\lambda \in L(M, m)$ into two parts, i.e. we write $\lambda = \lambda' \& \lambda_M$ where $\lambda' \in L(M-1, m-1)$. This implies that

$$\sum_{\lambda} w(\lambda, \mathbf{k}) = \sum_{\lambda'} w(\lambda', \mathbf{k}') \sum_{k=\lambda'_{M-1}}^{m-1} w^*(\mathbf{k}, k+1)$$

where

$$w^*(\mathbf{k}, k) := (n - (k - M) - s_{M-1}(\mathbf{k}) - 1)^{\underline{\mathbf{k}_M - 1}}.$$

We note that the sum is of form

$$\sum_{k=\alpha_1}^{\alpha_2} (\alpha_3 - k)^{\underline{\alpha_4}}.$$

which can be computed easily if $\alpha_4 = \alpha_3 - \alpha_2$, since then

$$\sum_{k=\alpha_1}^{\alpha_2} (\alpha_3 - k)^{\underline{\alpha_4}} = \alpha_4! \binom{\alpha_3 - \alpha_1 + 1}{\alpha_4 + 1} = (\alpha_4 + 1)^{-1} (\alpha_3 - \alpha_1 + 1)^{\underline{\alpha_4 + 1}}$$

In this case the condition $\alpha_3 - \alpha_2 = \alpha_4$ is equivalent with

$$(n + M - s_{M-1}(\mathbf{k}) - 1) - m = \mathbf{k}_M - 1$$

This follows from the fact that $\mathbf{k} \in J_0(M, n - l)$ and $M > 1$, since this implies that

$$s_{M-1}(\mathbf{k}) = n - l - \mathbf{k}_M.$$

Therefore, the condition is equivalent with $M = m - l$ which holds by the assumption. We can combine the falling product to the last falling product in $w(\lambda', \mathbf{k}')$ which can be written as

$$w(\lambda'', \mathbf{k}'')(n - (\lambda'_{M-1} - (M - 1)) - s_{M-2}(\mathbf{k}) - 1)^{\underline{\mathbf{k}_{M-1}-1}}.$$

The last factor in this falling product is

$$J = (n - (\lambda'_{M-1} - (M - 1)) - s_{M-1}(\mathbf{k}) + 1)$$

since $s_{M-1} = s_{M-2} + \mathbf{k}_{M-1}$. On the other hand, the first factor in the falling product of w^* is

$$\alpha_3 - \alpha_1 + 1 = (n - (\lambda'_{M-1} - M) - s_{M-1}(\mathbf{k}) - 1) = J - 1.$$

Thus, the falling factors can be combined into a single falling factor of length $\mathbf{k}_M + \mathbf{k}_{M-1} - 1$ and the claim follows. \square

The last missing piece of the Section 3 is the proof of Lemma 3.7.

Proof of Lemma 3.7. We begin the proof with few observations and notations. First, let us start with fixed $\lambda \in L(m - l, m)$ and $\mathbf{k} \in J_0(m - l, n - l)$. We will denote the word $\pi(\mathbf{k}, \lambda)$ just by π for awhile. Let us denote the left-inverse of $j \mapsto \lambda_j$ by δ i.e. we define

$$\delta_j = \max\{k \mid \lambda_k \leq j\}.$$

We also observe that

$$\delta_j = \sum_{k=1}^j [\pi_k \neq 1]$$

which implies that

$$\sum_{k=1}^j [\pi_k = 1] = j - \delta_j.$$

Thus, for every j

$$s_j(\pi) = \sum_{k=1}^j [\pi_k = 1] + \sum_{k=1}^j \pi_k [\pi_k \neq 1] = (j - \delta_j) + s_{\delta_j}(\mathbf{k})$$

Since $\delta_{\lambda_j-1} = j - 1$, we obtain

$$s_{\lambda_j-1}(\pi) = (\lambda_j - 1 - (j - 1)) + s_{j-1}(\mathbf{k}).$$

Therefore, we have shown that

$$\prod_{j=1}^{m-l} (n - s_{\lambda_j-1}(\pi) - 1)^{\underline{\mathbf{k}_j-1}} = w(n, \lambda, \mathbf{k})$$

since $\psi(\mathbf{k}) = m - l$.

We can now sum over all λ 's and we obtain

$$\Lambda(n, \mathbf{k}) = \sum_{\lambda \in L(M, m)} w(n, \lambda, \mathbf{k})$$

and the claim follows from Lemma A.3. \square

APPENDIX B. PROOFS OF AUXILIARY RESULTS IN SECTION 4

We gather here the proofs of auxiliary lemmata we used in the Section 3. We start with Lemma 4.3 and Lemma 4.6 that deal with the matrices K_n and \tilde{K}_n .

Proof of Lemma 4.3. This follows by analysing the corresponding properties of the convolution operator $C(g)$ corresponding to a symbol g . For convolution operators, we can show that

$$C(g)C(h) = C(gh)$$

whenever $\hat{g} \asymp c_\alpha$, $\tilde{h} \asymp c_\beta$ and $\alpha + \beta < \frac{1}{2}$. Since the Fourier coefficients of g_α^{-1} (which follows Lemma 5.6) behave asymptotically as the Fourier coefficients of $g_{-\alpha}$, we deduce

$$C(g_\alpha)C(1/g_\alpha) = I$$

Expressing the convolution operator as an element of $\mathbb{R}^{\mathbb{Z} \times \mathbb{Z}}$ and dividing it into 9 blocks we have

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ A_{21} & A_{22} & A_{23} \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & B_{21}^* & \cdot \\ \cdot & B_{22} & \cdot \\ \cdot & B_{23}^* & \cdot \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & I_n & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

Since $A_{22} = T_n(g_\alpha)$ and $B_{22} = T_n(g_\alpha^{-1})$ we know A_{22} is invertible and therefore

$$B_{22} = A_{22}^{-1} - B_{22}(A_{21}B_{21}^* + A_{23}B_{23}^*)$$

If we denote $K_n(\alpha) = A_{21}B_{21}^* + A_{23}B_{23}^*$, the claim follows by symmetrizing the identity. \square

Proof of Lemma 4.6. We will drop subscript n from the following unless it is essential. We will denote the Fourier coefficients of g_α^{-1} by d_j and we will denote $c_j = c_\alpha(j)$. From the proof of Lemma 4.3 we know that

$$K_{ij} = \sum_{l=1}^{\infty} (c_{i+l}d_{j+l} + c_{n-i+l}d_{n-j+l}) = A_{ij} + A_{(n-i)(n-j)}.$$

Therefore, we may estimate

$$2\tilde{K}_{ij} \leq |B|_{ij} + |B|_{(n-i)(n-j)}$$

when $B = A + A^\top$. Since $|c_{i+l}| \asymp (i+l)^{2\alpha-1} \asymp (i \vee l)^{2\alpha-1}$ and analogously $|d_{j+l}| \asymp (j \vee l)^{-2\alpha-1}$, we can estimate

$$|B|_{ij} \lesssim n^{-1} \sum_{\pm} \int_{1/n}^{\infty} (x \vee t)^{-1 \pm 2\alpha} (t \vee y)^{-1 \mp 2\alpha} dt.$$

We notice that the right-hand side stays invariant in the transformations $\alpha \leftrightarrow -\alpha$ and $x \leftrightarrow y$, so we may assume that $\alpha = |\alpha| > 0$ and $x \leq y$. We will denote $v_t = t/y$ whenever $t \leq y$ and in particular, when $t = x$, we will denote $w = v_x$. Moreover, it holds that $v_t^{2\alpha} + v_t^{-2\alpha} \lesssim v_t^{-2\alpha}$ for all $t \leq y$.

When $t \leq x$ we have $t \vee x = x$ and $t \vee y = y$ and therefore

$$\sum_{\pm} \int_{1/n}^x \dots dt \lesssim y^{-1} (w^{2\alpha} + w^{-2\alpha}) \lesssim y^{-1} w^{-2\alpha}$$

When $x < t \leq y$, we have $t \vee x = t$ and $t \vee y = y$. Hence

$$\begin{aligned} \sum_{\pm} \int_x^y \dots dt &\lesssim y^{-1} \int_x^y t^{-1} (v_t^{2\alpha} + v_t^{-2\alpha}) dt \lesssim y^{-1} \int_x^y t^{-1} v_t^{-2\alpha} dt \\ &\lesssim y^{-1} w^{-2\alpha}. \end{aligned}$$

The remaining part has a trivial upper bound $2y^{-1} \lesssim y^{-1} w^{-2\alpha}$. Combining these three cases the claim follows. \square

Lemma 4.6 gives enough control for proving Proposition 4.5.

Proof of Proposition 4.5. Since α and β are fixed throughout the proof, we will drop them from the subscripts.

When $\alpha = 0$, the $\tilde{K}_n = 0$ and the claim is trivial. Therefore, we may suppose $\alpha \neq 0$ and since $\tilde{K}_n(\alpha)$ only depends on the absolute value of α , we may assume $\alpha > 0$ as well.

When $\beta = 0$, the Toeplitz matrix $T_n = I$. Therefore,

$$|T_n| : \tilde{K}_n \asymp \int_{1/n}^{1-1/n} k(x, x) dx \asymp \int_{1/n}^1 x^{-1} dx \asymp \log n$$

which implies the claim in this case. So we may assume that $\beta \neq 0$ in the sequel.

We have an asymptotic representation

$$(37) \quad |T_n|_{ij} \asymp n^{-1} |x - y|^{2\beta-1} [|x - y| > n^{-1}] + [i = j].$$

Since we already computed the claim for identity matrix, we may concentrate to contribution coming from outside the main diagonal.

By Lemma 4.6, the representation (37) and the symmetry $(x, y) \leftrightarrow (1-x, 1-y)$ we see that

$$|T_n| : \tilde{K}_n \lesssim n^{2\beta} \int_{I_n} \frac{(x \vee y)^{-1+2\alpha}}{(x \wedge y)^{2\alpha}} |x - y|^{-1+2\beta} dx dy = n^{2\beta} \int_{I_n} f.$$

where $I_n = \{|x - y| > n^{-1}, x \wedge y \geq n^{-1}\}$. Let us keep y fixed first. If we suppose $y/2 < x < 2y$, we have an estimate $x \wedge y \asymp x \vee y \asymp y$. Therefore, we have

$$\int_{y/2}^{2y} f(x, y) [(x, y) \in I_n] dx \asymp y^{-1} \int_{1/n}^{y \wedge (1-y)} x^{2\beta-1} dx$$

Considering the cases $y < \frac{1}{2}$ and $y \geq \frac{1}{2}$ separately, we obtain

$$\int_0^1 dy \int_{y/2}^{2y} f(x, y) [(x, y) \in I_n] dx \asymp n^{-2\beta} \log n [\beta < 0] + [\beta > 0].$$

When $x \leq y/2$, we have an estimate $|x - y| \asymp y$. In this case the integral reduces to

$$\int_0^{y/2} f(x, y) [(x, y) \in I_n] dx dy \asymp y^{-1+2\beta}$$

since $\alpha < \frac{1}{2}$. When $x \geq 2y$, we have an estimate $|x - y| \asymp x$. In this case the integral can therefore be estimated as

$$\int_{2y}^1 f(x, y) [(x, y) \in I_n] dx dy \asymp [y < \frac{1}{2}] (y^{-1+2\beta} \vee y^{-2\alpha}).$$

Integrating these two last cases with respect to y and summing all the cases together shows that

$$n^{2\beta} \int_{I_n} f(x, y) dx dy \asymp \log n [\beta < 0] + n^{2\beta} [\beta > 0] \asymp n^{2\beta} \vee \log n$$

and the claim follows. \square

Proof of Lemma 4.7. By Lemma 5.3 and the reasoning explained in Section 5 we know that $|T_n(g_{-\alpha})^{-1}|$ behaves elementwise as $|T_n(\theta_{2\alpha})^{-1}|$. The diagonal estimate follows from [26, Théorème 1]. Outside the diagonal, we divide the proof in two parts $\alpha > 0$ and $\alpha < 0$. Since x and y will be fixed throughout the proof, we will usually drop them from parameters of functions for notational simplicity.

When $\alpha > 0$, we have

$$T_{ij}^{-1} \asymp n^{-1+2\alpha} \mathcal{S}(f)(x, y)$$

where the function f in the triangle $x \vee \tilde{x} \leq y < 1$ is given by

$$f(x, y) = x^\alpha y^\alpha \int_y^1 \rho(t) t^{-2\alpha} dt \asymp x^\alpha \int_y^1 \rho(t) dt.$$

Here and later we will denote

$$\rho(t) = (t - x)^{\alpha-1} (t - y)^{\alpha-1}.$$

The function ρ satisfies

$$\rho(t) \asymp [t < z] (t - y)^{\alpha-1} w^{\alpha-1} + [t \geq z] (t - y)^{2\alpha-2}$$

where $z = 2y - x$ and $w = y - x$. When $y \in I_b(x)$ we have $[t \geq z] = 0$ and therefore,

$$f \asymp w^{\alpha-1} x^\alpha \tilde{y}^\alpha = E_2^{(\alpha)}.$$

When $y \in I_d(x)$, we have $z \leq 1$ and in this case

$$\int_y^1 \rho(t) dt \asymp w^{\alpha-1} \int_0^w t^{\alpha-1} dt + \int_w^{\tilde{y}} t^{2\alpha-2} dt \asymp w^{2\alpha-1},$$

giving the claim for $\alpha > 0$.

When $\alpha < 0$, we similarly have

$$T_{ij}^{-1} \asymp n^{-1+2\alpha} \mathcal{S}(f + f_2)(x, y)$$

where the functions f and f_2 in the triangle $x \vee \tilde{x} \leq y < 1$ are given by

$$f(x, y) = -x^\alpha y^\alpha \int_y^1 \rho(t) (\rho_2(t) - \rho_2(y)) t^{-2\alpha} dt$$

$$\text{and } f_2(x, y) = \alpha^{-1} x^\alpha \tilde{y}^\alpha y^{-\alpha} w^{\alpha-1} \asymp -x^\alpha \tilde{y}^\alpha w^{\alpha-1}.$$

The auxiliary function ρ_2 is given by

$$\rho_2(s) = \left(\frac{w}{s-x} \right)^{\alpha-1} \left(\frac{s}{y} \right)^{2\alpha}.$$

We note that

$$(38) \quad \begin{aligned} \rho_2'(s) &= ((\alpha+1)s - 2\alpha x) s^{-1} (s-x)^{-1} \rho_2(s) \asymp w^{\alpha-1} (s-x)^{-\alpha} \\ &\asymp [s < z] w^{-1} + [s \geq z] w^{\alpha-1} (s-y)^{-\alpha}. \end{aligned}$$

When $y \in I_b(x)$ we have $[s \geq z] = [t \geq z] = 0$ and therefore,

$$\rho(t) (\rho_2(t) - \rho_2(y)) t^{-2\alpha} \asymp \rho(t) \int_y^t w^{-1} ds \asymp (t-y)^\alpha w^{\alpha-2}$$

and hence for every $y \in I_b(x)$ we have

$$0 \leq -f(x, y) \lesssim x^\alpha \tilde{y}^{\alpha+1} w^{\alpha-2} \lesssim -f_2(x, y)$$

where in the last estimate we used the fact that $\tilde{y} w^{-1} \leq 1$ if and only if $y \in I_b(x)$. Thus, $f + f_2 \asymp -E_2^{(\alpha)}$ whenever $y \in I_b(x)$.

When $y \in I_d(x)$ we have more cases. First we note that $[s < z] \geq [t < z]$ and therefore,

$$[t < z] \rho(t) (\rho_2(t) - \rho_2(y)) t^{-2\alpha} \asymp [t < z] (t-y)^\alpha w^{\alpha-2}$$

This leads to

$$\int_y^z \rho(t) (\rho_2(t) - \rho_2(y)) t^{-2\alpha} dt \asymp w^{-1+2\alpha}$$

When $t \geq z$, we use a cruder estimate of ρ_2' by estimating the indicators functions on the right hand side above by constants which gives an estimate

$$0 \leq \rho(t) (\rho_2(t) - \rho_2(y)) t^{-2\alpha} \lesssim w^{-1} (t-y)^{2\alpha-1} + w^{\alpha-1} (t-y)^{\alpha-1}$$

Since $(t - y)^\alpha \leq w^\alpha$ this implies that

$$0 \leq \int_z^1 \rho(t)(\rho_2(t) - \rho_2(y))t^{-2\alpha} dt \lesssim w^{\alpha-1} \int_w^{\tilde{y}} t^{\alpha-1} dt \lesssim w^{-1+2\alpha}$$

Therefore, $f(x, y) \asymp -x^\alpha w^{-1+2\alpha}$. Since

$$0 \leq -f_2 \lesssim (\tilde{y}w^{-1})^\alpha f \leq f$$

whenever $y \in I_d(x)$ the claim follows. \square

Lemma B.1. *Let $\alpha, \beta \in (-\frac{1}{2}, \frac{1}{2})$ and denote $\nu = \alpha_- + \beta_+$. We have the following asymptotic estimates:*

– For every $\gamma = 2\beta \neq 1$ it holds

$$(A) \quad n^\gamma \int_{I_n^2} |x_1 - x_2|_n^{-2+\gamma} dx_1 dx_2 \asymp n^{\gamma \vee 1}$$

– For every $\alpha \neq 0$ and $\beta \neq 0$ it holds

$$(B) \quad \int_{I_n^2} |x_1 - x_2|_n^{-1+2\beta} x_1^{-1-\alpha} dx_1 dx_2 \asymp n^{2\beta_- + \alpha_+}$$

– For every $x_1, x_3 \in I_n$ such that $x_3 - x_1 > 3n^{-1}$ and for $\gamma = \nu - \frac{1}{2}$ it holds

$$(C) \quad \int_{n^{-1}}^{\frac{x_1+x_3}{2}} |x_1 - x_2|_n^{-1+2\beta} |x_2 - x_3|^{-1-2\alpha} dx_2 \\ \asymp [\gamma < 0] n^{2\beta_-} |x_1 - x_3|_n^{-1+2(\beta_+ - \alpha)} + [\gamma > 0] x_1^{2\gamma}$$

Proof of Proposition 4.2. In order to prove the claim, we first use the representation of the inverse matrix $T_n(g_\alpha)^{-1}$ from Lemma 4.7 and use the triangle inequality for norms to conclude that

$$\begin{aligned} & \|T_n(g_\beta)^{1/2} T_n(g_\alpha)^{-1} T_n(g_\beta)^{1/2}\|_F \\ & \asymp \|T_n(g_\beta)^{1/2} (I + \tilde{E}_1^{(-\alpha, n)} + \tilde{E}_2^{(-\alpha)}) T_n(g_\beta)^{1/2}\|_F \\ & \leq \|T_n(g_\beta)\|_F + \|T_n(g_\beta)^{1/2} \tilde{E}_1^{(-\alpha, n)} T_n(g_\beta)^{1/2}\|_F \\ & \quad + \|T_n(g_\beta)^{1/2} \tilde{E}_2^{(-\alpha)} T_n(g_\beta)^{1/2}\|_F \end{aligned}$$

The first term on the right-hand side is handled directly with the estimate (A) from Lemma B.1 since

$$\|T_n(g_\beta)\|_F^2 \asymp n^{4\beta} \int_{I_n^2} |x - y|^{-2+4\beta} dx dy \asymp n^{4\beta \vee 1}$$

when $\beta \neq \frac{1}{4}$.

The second term can be estimated with the help of Lemma B.2 which yields that

$$\|T_n(g_\beta)^{1/2} \tilde{E}_1^{(-\alpha, n)} T_n(g_\beta)^{1/2}\|_F \asymp n^{2(\alpha_- + \beta_+) \vee 1/2}$$

This already implies that the sum of the first two terms gives the claimed asymptotics. We still need to show that the third term has at most the claimed growth properties but in order to do that we need to split the symbol into two parts and we do this by estimating

$$\begin{aligned} \left\| T_n(g_\beta)^{1/2} \tilde{E}_2^{(-\alpha)} T_n(g_\beta)^{1/2} \right\|_F &\leq \left\| T_n(g_\beta)^{1/2} \tilde{E}_3^{(-\alpha)} T_n(g_\beta)^{1/2} \right\|_F \\ &\quad + \left\| T_n(g_\beta)^{1/2} \tilde{E}_4^{(-\alpha)} T_n(g_\beta)^{1/2} \right\|_F \end{aligned}$$

Here $\tilde{E}_3^{(-\alpha)}$ and $\tilde{E}_4^{(-\alpha)}$ are $n \times n$ -matrices corresponding to kernels $\mathcal{S}(E_2^{(-\alpha)} [x \leq \frac{2}{3}])$ and $\mathcal{S}(E_2^{(-\alpha)} [x > \frac{2}{3}])$, respectively. Therefore, using Lemmata B.3 and B.4 the claim follows. \square

Proof of Proposition 4.4. Throughout the proof we assume $\alpha \neq 0$. The proof of this estimate follows from the following estimates

$$(39) \quad I\tilde{K}_n(\alpha) \lesssim n^{2\alpha-} \tilde{K}_n(\alpha)$$

$$(40) \quad \begin{aligned} n^{-1-2\alpha} \int_{n^{-1}}^{1-n^{-1}} |x-t|_n^{-1-2\alpha} k_\alpha(t, y) [t \in \mathcal{S}(I_d(x))] dt \\ \asymp n^{-1-2\alpha-} k_\alpha(x, y) \end{aligned}$$

and

$$(41) \quad \begin{aligned} n^{-1-2\alpha} \int_{n^{-1}}^{1-n^{-1}} \mathcal{S}(E_2^\alpha)(x, t) k_\alpha(t, y) [t \notin \mathcal{S}(I_d(x))] dt \\ \lesssim n^{-1-2\alpha-} k_\alpha(x, y). \end{aligned}$$

Summing these estimates together with the representation of the matrix $|T_n(g_{-\alpha})^{-1}|$ given by Lemma 4.7 implies the claim. Moreover, we notice from the asymptotic estimate (40) that the estimate is actually sharp and we the same estimate for the lower bound.

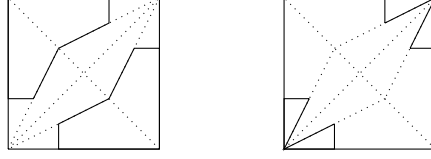
The first estimate (39) is trivial and the latter two estimates (40) and (41) are given by the auxiliary lemmata B.5 and B.6 respectively. \square

Lemma B.2. *For every $\alpha, \beta \in (-1/2, 1/2)$ we have that*

$$\left\| T_n(g_\beta)^{1/2} \tilde{E}_1^{(-\alpha, n)} T_n(g_\beta)^{1/2} \right\|_F \asymp n^{2(\alpha_- + \beta_+) \vee 1/2}$$

Proof. The squared norm has an asymptotic estimate

$$\begin{aligned} \left\| T_n(g_\beta)^{1/2} \tilde{E}_1^{(-\alpha, n)} T_n(g_\beta)^{1/2} \right\|_F^2 &\asymp n^{4(\beta-\alpha)} \int_{I_n^4} |x_1 - x_2|_n^{-1+2\beta} \times \\ &\quad \times |x_2 - x_3|_n^{-1-2\alpha} |x_3 - x_4|_n^{-1+2\beta} |x_4 - x_1|_n^{-1-2\alpha} dx_1 \dots dx_4 \end{aligned}$$

FIGURE 2. Supports of the kernels $E_3^{(-\alpha)}$ and $E_4^{(-\alpha)}$

We can use Lemmata B.12 and B.13 to conclude that the integral on the right-hand side with respect to x_3 is

$$\begin{aligned} \int_{I_n^1} \dots dx_3 &= \int_{I_n^1} ([|x_2 - x_4| > 3n^{-1}] + [|x_2 - x_4| \leq 3n^{-1}]) \dots dx_3 \\ &\asymp [\gamma < 0] |x_2 - x_4|^{-1+2(\beta-\alpha)} \sum_{\rho \in \{\alpha_+, \beta_-\}} n^{2\rho} |x_2 - x_4|^{2\rho} \\ &\quad + [\gamma > 0] (x_2 \wedge x_4)^{2\gamma}. \end{aligned}$$

where $\gamma = \alpha_- + \beta_+ - 1/2$. We can repeat this integration with respect to x_1 and combining these we obtain that

$$\begin{aligned} \int_{I_n^4} \dots dx &\asymp [\gamma < 0] \sum_{\rho_1, \rho_2 \in \{\alpha_+, \beta_-\}} n^{2(\rho_1 + \rho_2)} \int_{I_n^2} |x_2 - x_4|^{-2+4(\beta-\alpha)+2(\rho_1 + \rho_2)} dx \\ &\quad + [\gamma > 0] \int_{I_n^2} |x_2 - x_4|^{-2+4(\beta-\alpha)} (x_2 \wedge x_4)^{4\gamma} dx \end{aligned}$$

Supposing $\gamma > 0$. Then using the (B) from Lemma B.1 we have an upper estimate

$$n^{4(\beta-\alpha)} \int_{I_n^2} |x_2 - x_4|^{-2+4(\beta-\alpha)} (x_2 \wedge x_4)^{4\gamma} dx \lesssim n^{4(\beta_+ + \alpha_-)}$$

and we can easily deduce that the estimate holds also from below. When $\gamma < 0$, we can apply (A) from Lemma B.1 to conclude that

$$n^{4(\beta-\alpha)} \int_{I_n^2} \dots \asymp \sum_{\rho_1, \rho_2 \in \{\alpha_+, \beta_-\}} n^{(4(\beta-\alpha)+2(\rho_1 + \rho_2)) \vee 1}$$

and it is straightforward to verify that this is $\asymp n^{4(\beta_+ + \alpha_-) \vee 1}$. Combining both cases and taking the square root implies the claim. \square

Lemma B.3. *For every $\alpha, \beta \in (-1/2, 1/2)$ we have that*

$$\left\| T_n(g_\beta)^{1/2} \widetilde{E}_3^{(-\alpha)} T_n(g_\beta)^{1/2} \right\|_F \lesssim n^{2(\alpha_- + \beta_+) \vee 1/2}$$

where the kernel of $E_3^{(-\alpha)}$ is $\mathcal{S}(E_2^{(-\alpha)} [x \leq \frac{2}{3}])$.

Proof. We can use the rather trivial estimate

$$\mathcal{S}(E_2^{(-\alpha)} [x \leq \frac{2}{3}]) \lesssim n^{2\alpha_+}$$

which follows since $E_2^{(-\alpha)}(x, y) \lesssim n^{2\alpha_+}$ in when $x \leq \frac{2}{3}$. This implies that

$$\begin{aligned} & \left\| T_n(g_\beta)^{1/2} \tilde{E}_3^{(-\alpha)} T_n(g_\beta)^{1/2} \right\|_F^2 \\ & \lesssim n^{4(\beta-\alpha)} n^{4\alpha_+} \int_{I_n^4} |x_1 - x_2|_n^{-1+2\beta} |x_3 - x_4|_n^{-1+2\beta} dx_1 \dots dx_4 \\ & \lesssim n^{4(\beta+\alpha_-)} \left(\int_{I_n^2} |x_1 - x_2|_n^{-1+2\beta} dx_1 dx_2 \right)^2 \end{aligned}$$

The integral on the right-hand side is a special case of (B) in Lemma B.1 when $\alpha = -1$. Therefore,

$$\begin{aligned} & \left\| T_n(g_\beta)^{1/2} \tilde{E}_3^{(-\alpha)} T_n(g_\beta)^{1/2} \right\|_F^2 \\ & \lesssim n^{4(\beta+\alpha_-)} n^{4\beta_-} = n^{4(\beta_+ + \alpha_-)} \end{aligned}$$

and the claim follows. \square

Lemma B.4. *For every $\alpha, \beta \in (-1/2, 1/2)$ we have that*

$$\left\| T_n(g_\beta)^{1/2} \tilde{E}_4^{(-\alpha)} T_n(g_\beta)^{1/2} \right\|_F \lesssim n^{2(\alpha_- + \beta_+)}$$

where the kernel of $E_4^{(-\alpha)}$ is $\mathcal{S}(E_2^{(-\alpha)} [x > \frac{2}{3}])$.

Proof. We can use a similar estimate as in Lemma B.3 but this time we cannot estimate the indicator function with a constant. Thus, we use an estimate

$$\mathcal{S}(E_2^{(-\alpha)} [x > \frac{2}{3}]) \lesssim n^{\alpha_+} \mathcal{S}([x > \frac{2}{3}] \tilde{x}^{-1-\alpha})$$

which follows since $E_2^{(-\alpha)}(x, y) \lesssim n^{\alpha_+} |x - y|^{-1-\alpha}$ in when $x > \frac{2}{3}$ and moreover we can estimate $|x - y|^{-1-\alpha} \lesssim \tilde{x}^{-1-\alpha}$ given $x > \frac{2}{3}$. We can divide the support into the four pieces and denote them by $J_{4,1}, \dots, J_{4,4}$ and then express the right-hand side as a sum

$$\begin{aligned} & \mathcal{S}([x > \frac{2}{3}] \tilde{x}^{-1-\alpha}) \\ & = \tilde{x}^{-1-\alpha} [J_{4,1}] + \tilde{y}^{-1-\alpha} [J_{4,2}] + x^{-1-\alpha} [J_{4,3}] + y^{-1-\alpha} [J_{4,4}] \\ & =: I_1 + \dots + I_4 \end{aligned}$$

Therefore, the squared norm can be estimated as

$$\begin{aligned} & \left\| T_n(g_\beta)^{1/2} \tilde{E}_4^{(-\alpha)} T_n(g_\beta)^{1/2} \right\|_F^2 n^{-4(\beta-\alpha)-2\alpha_+} \\ & \lesssim \sum_{j,k=1}^4 \int_{I_n^4} |x_1 - x_2|^{-1+2\beta} I_j(x_2, x_3) |x_3 - x_4|^{-1+2\beta} I_k(x_4, x_1) dx \\ & = 4 \sum_{j=1}^4 \int_{I_n^4} |x_1 - x_2|^{-1+2\beta} I_3(x_2, x_3) |x_3 - x_4|^{-1+2\beta} I_j(x_4, x_1) dx \end{aligned}$$

where the last identity follows by using the symmetries $(x_j)_j \leftrightarrow (\tilde{x}_j)_j$ and $(x_j)_j \leftrightarrow (x_{5-j})_j$. The terms in the sum are essentially of two types, which we can call *evenly* (when $j \in \{1, 3\}$) and *unevenly bound* (when $j \in \{2, 4\}$).

The evenly bound terms are easier, since we don't need the indicators any more and we can use estimates $I_3(x, y) \lesssim x^{-1-\alpha}$ and $I_1(x, y) \lesssim \tilde{x}^{-1-\alpha}$. This implies that for the first evenly bound case ($j = 3$) we have

$$\begin{aligned} & \int_{I_n^4} |x_1 - x_2|^{-1+2\beta} I_3(x_2, x_3) |x_3 - x_4|^{-1+2\beta} I_3(x_4, x_1) dx \\ & \lesssim \left(\int_{I_n^2} |x_1 - x_2|^{-1+2\beta} x_2^{-1-\alpha} dx_1 dx_2 \right)^2 \asymp n^{4\beta_- + 2\alpha_+} \end{aligned}$$

where the last estimate follows directly from (B) in Lemma B.1. The second evenly bound case needs one extra application of symmetry $(x_3, x_4) \leftrightarrow (\tilde{x}_1, \tilde{x}_2)$ after the four-dimensional integral has been split to a product of two two-dimensional integrals. Thus, also the second evenly bound case has exactly the estimate, namely

$$\int_{I_n^4} |x_1 - x_2|^{-1+2\beta} I_3(x_2, x_3) |x_3 - x_4|^{-1+2\beta} I_1(x_4, x_1) dx \lesssim n^{4\beta_- + 2\alpha_+}$$

For estimating the unevenly bound cases ($j \in \{2, 4\}$) we have to take the indicator functions into account. In both cases we can first integrate with respect to x_3 . The function depending on x_3 in both cases is $[(x_2, x_3) \in J_{4,3}] |x_3 - x_4|^{-1+2\beta}$. This can be easily estimated

$$\int_{I_n^1} [(x_2, x_3) \in J_{4,3}] |x_3 - x_4|^{-1+2\beta} dx_3 \lesssim n^{2\beta_-}$$

Next we integrate with respect to x_4 . The reminding part depending on x_4 is just the indicator function $[(x_4, x_1) \in J_{4,j}]$. In the first unevenly bound case ($j = 4$) we can estimate this by

$$\int_{I_n^1} [(x_4, x_1) \in J_{4,4}] dx_4 \leq \int_{I_n^1} [x_4 \leq \tfrac{1}{2}x_1] dx_4 \lesssim x_1$$

This means that we have an upper estimate

$$\begin{aligned} & \int_{I_n^4} |x_1 - x_2|^{-1+2\beta} I_3(x_2, x_3) |x_3 - x_4|^{-1+2\beta} I_4(x_4, x_1) dx \\ & \lesssim n^{2\beta_-} \int_{I_n^2} x_1^{-\alpha} x_2^{-1-\alpha} |x_1 - x_2|^{-1+2\beta} dx_1 dx_2. \end{aligned}$$

The singularity at $x_1 = 0$ is integrable, so we first integrate with respect to x_1 . We can split the integration into two parts

$$\begin{aligned} \int_{I_n^1} x_1^{-\alpha} |x_1 - x_2|^{-1+2\beta} dx_1 &= \int_{I_n^{-1}} ([x_1 \leq \tfrac{1}{2}x_2] + [x_1 > \tfrac{1}{2}x_2]) \dots \\ &\lesssim x_2^{2\beta-\alpha} + n^{2\beta_-} x_2^{-\alpha} \lesssim n^{2\beta_-} x_2^{-\alpha} \end{aligned}$$

and hence

$$\begin{aligned} n^{2\beta_-} \int_{I_n^2} x_1^{-\alpha} x_2^{-1-\alpha} |x_1 - x_2|^{-1+2\beta} dx_1 dx_2 &\lesssim n^{4\beta_-} \int_{I_n^1} x_2^{-1-2\alpha} dx_2 \\ &\asymp n^{4\beta_-+2\alpha_+} \end{aligned}$$

Similarly, the second unevenly bound case can be estimated to give

$$\int_{I_n^1} [(x_4, x_1) \in J_{4,2}] dx_4 \lesssim \tilde{x}_1$$

This means that

$$\begin{aligned} &\int_{I_n^4} |x_1 - x_2|^{-1+2\beta} I_3(x_2, x_3) |x_3 - x_4|^{-1+2\beta} I_2(x_4, x_1) dx \\ &\lesssim n^{2\beta_-} \int_{I_n^2} \tilde{x}_1^{-\alpha} x_2^{-1-\alpha} |x_1 - x_2|^{-1+2\beta} dx_1 dx_2. \end{aligned}$$

By symmetry, the integral with respect to x_1 has an estimate

$$\int_{I_n^1} \tilde{x}_1^{-\alpha} |x_1 - x_2|^{-1+2\beta} dx_1 \lesssim n^{2\beta_-} \tilde{x}_2^{-\alpha}$$

which means that the singularity is split in two parts and we obtain

$$\begin{aligned} n^{2\beta_-} \int_{I_n^2} \tilde{x}_1^{-\alpha} x_2^{-1-\alpha} |x_1 - x_2|^{-1+2\beta} dx_1 dx_2 &\lesssim n^{4\beta_-} \int_{I_n^1} x_2^{-1-\alpha} \tilde{x}_2^{-\alpha} dx_2 \\ &\asymp n^{4\beta_-+\alpha_+} \end{aligned}$$

Therefore, when we combine all the previous estimates we obtain the claimed estimate for the squared norm

$$\left\| T_n(g_\beta)^{1/2} \tilde{E}_4^{(-\alpha)} T_n(g_\beta)^{1/2} \right\|_F^2 \lesssim n^{4\beta_-+2\alpha_+} n^{4(\beta_--\alpha)+2\alpha_+} = n^{4(\beta_++\alpha_-)}.$$

□

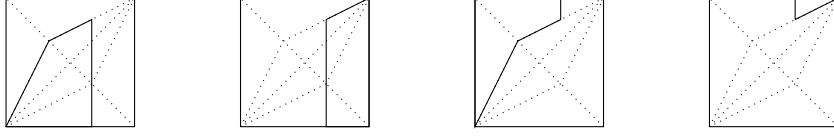
Lemma B.5. *The estimate (40) holds for every $0 \neq \alpha \in (-\frac{1}{2}, \frac{1}{2})$.*

Proof. Suppose $y \notin \mathcal{S}(I_d(x))$. Then we have $k_\alpha(t, y) \asymp k_\alpha(x, y)$ for every $t \in \mathcal{S}(I_d(x))$ uniformly in t .

When $y \in \mathcal{S}(I_d(x))$ we have that $k_\alpha(t, y) \asymp k_\alpha(x, x)$ uniformly in $t \in \mathcal{S}(I_d(x))$. Therefore, $k_\alpha(t, y) \asymp k_\alpha(x, y)$ uniformly for every $t \in \mathcal{S}(I_d(x))$. This implies that

$$\begin{aligned} &\int_{n^{-1}}^{1-n^{-1}} |x - t|_n^{-1-2\alpha} k_\alpha(t, y) [t \in \mathcal{S}(I_d(x))] dt \\ (42) \quad &\asymp k_\alpha(x, y) \int_{n^{-1}}^{1-n^{-1}} |x - t|_n^{-1-2\alpha} [t \in \mathcal{S}(I_d(x))] dt \\ &\asymp k_\alpha(x, y) (n^{2\alpha} [\alpha > 0] + [\alpha < 0]) \end{aligned}$$

where the last estimate follows by direct integration. This implies the claim. □

FIGURE 3. Illustration of indicators of J_j

Lemma B.6. *The estimate (41) holds for every $0 \neq \alpha \in (-\frac{1}{2}, \frac{1}{2})$.*

Proof. In order to obtain the estimate (41), we divide the integration set $\{t \notin \mathcal{S}(I_d(x))\}$ into *lower* and *upper* parts, where an element $t \notin \mathcal{S}(I_d(x))$ belongs to $t \in \text{lower}$ when $n^{-1} < t < x$ and $t \in \text{upper}$ when $x < t < 1 - n^{-1}$. Therefore, we can write

$$\int_{n^{-1}}^{1-n^{-1}} \dots [t \notin \mathcal{S}(I_d(x))] dt = \int_{\text{lower}} + \int_{\text{upper}} \dots dt$$

We can exploit the symmetry $k_\alpha(x, y) = k_\alpha(\tilde{x}, \tilde{y})$ and $|t - x| = |\tilde{t} - \tilde{x}|$ together with change of variables that to reduce showing that the estimate

$$(43) \quad \int_{\text{lower}} \mathcal{S}(E_2^\alpha)(x, t) k_\alpha(t, y) dt \lesssim n^{2\alpha+} k_\alpha(x, y).$$

holds for every x, y for the integral over the lower interval.

This in turn is obtained by showing the estimate by assuming in addition that $(x, y) \in J_1, J_2, J_3$ or $(x, y) \in J_4$ where

$$(44) \quad \begin{aligned} J_1 &:= \{ (x, y) \mid x < \frac{2}{3}, y \in \mathcal{S}(I_d(x)) \text{ or } x < \frac{2}{3} \wedge y \} \\ J_2 &:= \{ (x, y) \mid \frac{2}{3} \leq x < 1, y \in \mathcal{S}(I_d(x)) \text{ or } \frac{2}{3} \leq x < y \} \\ J_3 &:= \{ (x, y) \in \mathcal{S}(I_b) \mid y < x < \frac{2}{3} \} \\ J_4 &:= \{ (x, y) \in \mathcal{S}(I_b) \mid x > y \vee \frac{2}{3} \} \end{aligned}$$

Since $\{J_1, \dots, J_4\}$ is a partition of the unit square, these together yield the claim. These estimates follow from Lemmata B.7, B.8, B.9 and B.10, respectively. \square

Lemma B.7. *Suppose $(x, y) \in J_1$ where J_1 is defined as in (44). Then the estimate (43) holds for every $\alpha \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$.*

Proof. In this case, we notice that we can estimate the integral on the left-hand side of (43) by

$$\int_{\text{lower}} \dots dt \asymp x^{-1-\alpha} \int_{n^{-1}}^{x/2} (t^{-\alpha-2|\alpha|} y^{-1+2|\alpha|} + t^{-\alpha} \tilde{y}^{-2|\alpha|}) dt$$

In order to estimate this we divide this into three parts $\{\alpha < 0\}$, $\{0 < \alpha < \frac{1}{3}\}$ and $\{\frac{1}{3} < \alpha < \frac{1}{2}\}$. Therefore, by direct integration we

obtain an estimate

$$\begin{aligned} \int_{\text{lower}} \dots dt &\asymp [\alpha < 0] y^{-1+2|\alpha|} + [0 < \alpha < \frac{1}{3}] (x^{-4|\alpha|} y^{-1+2|\alpha|}) \\ &\quad + [\frac{1}{3} < \alpha < \frac{1}{2}] x^{-1-\alpha} n^{3\alpha-1} y^{-1+2|\alpha|} + x^{-2\alpha} \tilde{y}^{-2|\alpha|} \end{aligned}$$

In this region $k_\alpha(x, y) \asymp x^{-2|\alpha|} y^{-1+2|\alpha|} + \tilde{y}^{-2|\alpha|}$ and the claim follows, since every term can be bounded from above by $n^{2\alpha+} k_\alpha(x, y)$. \square

Lemma B.8. *Suppose $(x, y) \in J_2$ where J_2 is defined as in (44). Then the estimate (43) holds for every $\alpha \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$.*

Proof. In this case we have an estimate $k_\alpha(x, y) \asymp \tilde{x}^{-1+2|\alpha|} \tilde{y}^{-2|\alpha|}$. Moreover, the integral can be estimated as

$$\begin{aligned} \int_{\text{lower}} \dots dt &\asymp \tilde{x}^{-\alpha} \int_{n^{-1}}^{2x-1} (x-t)^{-1-\alpha} t^{-\alpha-2|\alpha|} dt \\ &\quad + \tilde{x}^{-\alpha} \tilde{y}^{-2|\alpha|} \int_{n^{-1}}^{2x-1} (x-t)^{-1-\alpha} t^{-\alpha} \tilde{t}^{-1+2|\alpha|} dt \end{aligned}$$

Let's denote the right-hand side as $I_1 + I_2$. When $\alpha < 0$, the I_1 can be easily estimated, since then

$$\tilde{x}^{-\alpha} \int_{n^{-1}}^{x/2} + \int_{x/2}^{2x-1} \dots \asymp \tilde{x}^{|\alpha|} \left(1 + \int_{x/2}^{2x-1} (x-t)^{-1+|\alpha|} dt \right) \lesssim k_\alpha(x, y)$$

When $\alpha < 0$, the part I_2 can be estimated from above as

$$\begin{aligned} I_2 &\lesssim \tilde{x}^{|\alpha|} \tilde{y}^{-2|\alpha|} \left(\int_{n^{-1}}^{x/2} t^{-\alpha} dt + \int_{x/2}^{2x-1} (x-t)^{-2+3|\alpha|} dt \right) \\ &\lesssim [\alpha < -\frac{1}{3}] \tilde{x}^{|\alpha|} \tilde{y}^{-2|\alpha|} + [-\frac{1}{3} < \alpha < 0] \tilde{x}^{-1+4|\alpha|} y^{-2|\alpha|} \\ &\lesssim k_\alpha(x, y) \end{aligned}$$

When $\alpha > 0$, the part I_2 can be estimated as

$$\begin{aligned} I_2 &\lesssim \tilde{x}^{-|\alpha|} \tilde{y}^{-2|\alpha|} \left(\int_{n^{-1}}^{x/2} t^{-\alpha} dt + \int_{x/2}^{2x-1} (x-t)^{-2+|\alpha|} dt \right) \\ &\lesssim \tilde{x}^{-1} \tilde{y}^{-2|\alpha|} \lesssim n^{2\alpha} k_\alpha(x, y) \end{aligned}$$

When $\alpha > 0$, the part I_1 estimate divides according to whether $\alpha > \frac{1}{3}$, $\alpha < \frac{1}{3}$ or $\alpha = \frac{1}{3}$. In two former cases $I_1 \lesssim k_\alpha(x, y)$ and in the last case $I_1 \lesssim \log n k_\alpha(x, y)$ which are all majorized by $n^{2\alpha} k_\alpha(x, y)$. In all cases we have an estimate

$$\begin{aligned} I_1 &\asymp \tilde{x}^{-2\alpha} + x^{-|\alpha|} \int_{n^{-1}}^{x/2} t^{-3\alpha} dt \\ &\lesssim k_\alpha(x, y) \left(\log n [0 < \alpha \leq \frac{1}{3}] + [\alpha > \frac{1}{3}] n^{3\alpha-1} \tilde{x}^{1-3\alpha} \right) \\ &\lesssim n^{2\alpha} k_\alpha(x, y) \end{aligned}$$

\square

Lemma B.9. *Suppose $(x, y) \in J_3$ where J_3 is defined as in (44). Then the estimate (43) holds for every $\alpha \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0, \frac{1}{3}\}$. When $\alpha \in \{0, \frac{1}{3}\}$ the estimate holds with logarithmic correction.*

Proof. In this case we have an estimate $k_\alpha(x, y) \asymp x^{-1+2|\alpha|}y^{-2|\alpha|}$. In this case, we divide the integration interval into two parts

$$\int_{\text{lower}} \dots dt = \int_{n^{-1}}^y + \int_y^{x/2} \dots dt =: I_1 + I_2.$$

The latter part I_2 is easier to estimate since

$$\begin{aligned} I_2 &\asymp x^{-1-\alpha}y^{-2|\alpha|} \int_y^{x/2} t^{-1-\alpha+2|\alpha|} dt \asymp k_\alpha(x, y)x^{-2\alpha} \\ &\lesssim n^{2\alpha+} k_\alpha(x, y) \end{aligned}$$

The former part I_1 needs bit more. We can estimate that

$$I_1 \asymp x^{-1-\alpha}y^{-1+2|\alpha|} \int_{n^{-1}}^y t^{-\alpha-2|\alpha|}$$

When $\alpha < 0$, we therefore have

$$I_1 \asymp x^{-1+|\alpha|}y^{|\alpha|} \asymp k_\alpha(x, y)x^{-|\alpha|}y^{3|\alpha|} \lesssim k_\alpha(x, y)$$

When $\alpha > 0$, then we have two cases $\alpha < \frac{1}{3}$ or $\frac{1}{3} < \alpha < \frac{1}{2}$. In the former we estimate

$$I_1 \asymp k_\alpha(x, y)x^{-3\alpha}y^\alpha \lesssim k_\alpha(x, y)x^{-2\alpha} \lesssim n^{2\alpha} k_\alpha(x, y)$$

and in the latter

$$I_1 \asymp k_\alpha(x, y)x^{-3\alpha}y^{-1+4\alpha}n^{3\alpha-1} \lesssim k_\alpha(x, y)x^{-1+\alpha}n^{3\alpha-1} \lesssim n^{2\alpha} k_\alpha(x, y)$$

□

Lemma B.10. *Suppose $(x, y) \in J_4$ where J_4 is defined as in (44) and suppose in addition that $y \leq \frac{1}{6}$. Then the estimate (43) holds for every $\alpha \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$.*

Proof. In this case $k_\alpha(x, y) \asymp y^{-2|\alpha|} + \tilde{x}^{-2|\alpha|}$ and we divide the integration interval into three parts

$$\int_{\text{lower}} \dots dt = \int_{n^{-1}}^y + \int_y^{1/6} + \int_{1/6}^{2x-1} \dots dt =: I_1 + I_2 + I_3.$$

The first integral can be estimated as

$$\begin{aligned} I_1 &\asymp \tilde{x}^{-\alpha} \int_{n^{-1}}^y t^{\alpha-2|\alpha|}y^{-1+2|\alpha|} + t^{-\alpha} dt \\ &\asymp \tilde{x}^{-\alpha} \left(\left[\alpha < \frac{1}{3} \right] y^{-\alpha} + \left[\alpha > \frac{1}{3} \right] n^{3\alpha-1}y^{-1+2\alpha} + y^{1-\alpha} \right) \\ &\lesssim k_\alpha(x, y) \left(\left[\alpha < \frac{1}{3} \right] + \left[\alpha > \frac{1}{3} \right] n^\alpha \right) \lesssim n^{\alpha+} k_\alpha(x, y) \end{aligned}$$

The estimation of the second integral is easier, since

$$I_2 \asymp \tilde{x}^{-\alpha} \int_y^{\frac{1}{6}} (t^{-1-\alpha+2|\alpha|} y^{-2|\alpha|} + t^{-\alpha}) dt$$

Now the antiderivative functions are increasing functions for every α and we have a constant upper integration upper bound and therefore,

$$I_2 \asymp \tilde{x}^{\mp|\alpha|} y^{-2|\alpha|} \lesssim n^{\alpha+} k_{\alpha}(x, y)$$

In the last integral I_3 we need to take into account the terms of form $(x-t)^{\gamma}$ but not the terms of form t^{γ} and so

$$I_3 \asymp \tilde{x}^{-\alpha} \int_{\frac{1}{6}}^{2x-1} (x-t)^{-1-\alpha} (y^{-2|\alpha|} + \tilde{t}^{-2|\alpha|}) dt$$

When $\alpha < 0$, we can therefore estimate that

$$\begin{aligned} I_3 &\asymp \tilde{x}^{|\alpha|} \left(y^{-2|\alpha|} + \int_{\frac{1}{6}}^{2x-1} (x-t)^{-1+|\alpha|} \tilde{t}^{-2|\alpha|} dt \right) \\ &\lesssim \tilde{x}^{|\alpha|} \left(y^{-2|\alpha|} + \tilde{x}^{-|\alpha|} \right) \lesssim k_{\alpha}(x, y) \end{aligned}$$

When $\alpha > 0$, we have

$$\begin{aligned} I_3 &\asymp \tilde{x}^{-2|\alpha|} y^{-2|\alpha|} + \tilde{x}^{-3|\alpha|} \int_{\frac{1}{6}}^{2x-1} (x-t)^{-1+|\alpha|} dt \\ &\lesssim n^{2\alpha} k_{\alpha}(x, y) + \tilde{x}^{-4|\alpha|} \lesssim n^{2\alpha} k_{\alpha}(x, y) \end{aligned}$$

since $\tilde{t}^{-2|\alpha|}$ behaves like $\tilde{x}^{-2|\alpha|}$ when t is near $2x-1$. Combining the estimates, we obtain the claim. \square

Lemma B.11. *Suppose $(x, y) \in J_4$ where J_4 is defined as in (44) suppose in addition that $y > \frac{1}{6}$. Then the estimate (43) holds for every $\alpha \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$.*

Proof. In this case $k_{\alpha}(x, y) \asymp \tilde{y}^{-1+2|\alpha|} \tilde{x}^{-2|\alpha|}$ and we divide the integration interval into three parts

$$\int_{\text{lower}} \dots dt = \int_{n^{-1}}^{1/6} + \int_{1/6}^y + \int_y^{2x-1} \dots dt =: I_1 + I_2 + I_3.$$

The first integral can be estimated as

$$I_1 \asymp \tilde{x}^{-\alpha} \left([\alpha < \frac{1}{3}] + [\alpha > \frac{1}{3}] n^{3\alpha-1} y^{-1+2\alpha} + \tilde{y}^{-2|\alpha|} \right)$$

Therefore, when $\alpha < 0$ we have $\tilde{x} \leq \tilde{y}$ and thus $I_1 \lesssim \tilde{x}^{-|\alpha|} \lesssim k_{\alpha}(x, y)$. When $0 < \alpha < \frac{1}{3}$, we can similarly estimate that $I_1 \lesssim k_{\alpha}(x, y) \tilde{y}^{1-3|\alpha|} \lesssim k_{\alpha}(x, y)$. The leading order singularity for I_1 comes when $\alpha > \frac{1}{3}$, where we have $I_1 \asymp k_{\alpha}(x, y) (\tilde{x}^{\alpha} n^{3\alpha-1} + \tilde{x}^{\alpha} \tilde{y}^{1-4\alpha}) \lesssim k_{\alpha}(x, y) n^{\alpha}$.

The second integral can be first estimated as

$$\begin{aligned} I_2 &\asymp \tilde{x}^{-\alpha} \int_{\frac{1}{6}}^y (x-t)^{-1-\alpha} (1 + \tilde{y}^{-2|\alpha|} \tilde{t}^{-1+2|\alpha|}) dt \\ &\lesssim \tilde{x}^{-\alpha} \tilde{y}^{-1} ([\alpha < 0] + [\alpha > 0] (x-y)^{-|\alpha|}) \\ &\lesssim k_\alpha(x, y) n^{2\alpha_+} \end{aligned}$$

where we also used the estimate $\tilde{t} \geq \tilde{y}$ and when $\alpha > 0$ we estimate $(x-y)^{-\alpha} \lesssim \tilde{x}^{-\alpha}$.

When $\alpha < 0$ the last integral I_3 can be estimated

$$\begin{aligned} I_3 &\asymp k_\alpha(x, y) \tilde{x}^{3|\alpha|} \int_y^{2x-1} (x-t)^{-1+|\alpha|} \tilde{t}^{-2|\alpha|} dt \\ &\lesssim k_\alpha(x, y) \tilde{x}^{2\alpha} \lesssim k_\alpha(x, y) \end{aligned}$$

and when $\alpha > 0$ we estimate

$$\begin{aligned} I_3 &\asymp k_\alpha(x, y) \tilde{x}^\alpha \int_y^{2x-1} (x-t)^{-1-\alpha} \tilde{t}^{-2\alpha} dt \\ &\lesssim k_\alpha(x, y) \tilde{x}^{-2\alpha} \lesssim k_\alpha(x, y) n^{2\alpha} \end{aligned}$$

Combining all the estimates, we obtain the claim. \square

Lemma B.12. *When $y > x + \frac{3}{n}$ and $\gamma := \alpha_- + \beta_+ - 1/2$ we have*

$$\begin{aligned} &\int_{n^{-1}}^{(x+y)/2} |x-t|_n^{-1-2\alpha} |y-t|_n^{-1+2\beta} dt \\ &\asymp [\gamma < 0] |y-x|_n^{-1+2(\beta-\alpha)} n^{2\beta_-} |y-x|_n^{2\beta_-} + [\gamma > 0] x^{2\gamma} \end{aligned}$$

Proof. This is (C) from Lemma B.1. \square

Lemma B.13. *When $y > x + \frac{3}{n}$ and $\gamma := \alpha_- + \beta_+ - 1/2$ we have*

$$\begin{aligned} &\int_{(x+y)/2}^{1-n^{-1}} |x-t|_n^{-1-2\alpha} |y-t|_n^{-1+2\beta} dt \\ &\asymp [\gamma < 0] |y-x|_n^{-1+2(\beta-\alpha)} n^{2\alpha_+} |y-x|_n^{2\alpha_+} + [\gamma > 0] x^{2\gamma} \end{aligned}$$

Proof. This follows from Lemma B.12 by denoting $y' := \tilde{x}$, $x' := \tilde{y}$, $\alpha' := -\beta$ and $\beta' := -\alpha$ and using change of variables $t' = 1-t$. \square

APPENDIX C. PROOFS OF AUXILIARY RESULTS IN SECTION 5

In this section we prove the technical results that were mentioned in Section 5. These augment the results of Rambour and Seghier [27, 26] to our setting.

Proof of Lemma 5.1. The proof of this was sketched already before the claim of Lemma 5.1, but let's provide some extra details. First let

$u \in H^2(\mathbb{D})$ be the unique solution of the Dirichlet problem for Laplace equation

$$\begin{cases} \Delta u = 0, & \text{in } \mathbb{D}, \\ u|_{\partial\mathbb{D}} = \frac{1}{2}\nu \end{cases}$$

where $\nu := \log f$. Let us define the analytic function $F = u + iv$. It is well known that the one harmonic conjugate v is obtained as a solution of

$$\begin{cases} \Delta v = 0, & \text{in } \mathbb{D}, \\ v|_{\partial\mathbb{D}} = \frac{1}{2}\mathcal{H}_0\nu. \end{cases}$$

All the others are of form $v + C$ for some constant $C \in \mathbb{C}$ since the Hilbert transform on the torus maps constants to zero. We can explicitly express the functions u and v and F in terms of the Fourier coefficients of $\nu := \log f$ on the boundary of the disk \mathbb{D} , namely

$$\begin{aligned} u(e^{it}) &= \sum_{k \in \mathbb{Z}} \frac{1}{2} \widehat{\nu}(k) e^{itk} \\ v(e^{it}) &= \sum_{k \in \mathbb{Z}} \frac{i}{2} \operatorname{sgn} k \widehat{\nu}(k) e^{itk}. \end{aligned}$$

Since $\nu \in L^1(\mathbb{T})$ the coefficients are bounded and go to zero and thus the analytic function F has a representation

$$F(z) = C + \frac{1}{2} \widehat{\nu}(0) + \sum_{k=1}^{\infty} \widehat{\nu}(k) z^k$$

For Grenander–Szegő result we need $F(0) = u(0)$ and since u has the sphere averaging property, we know that

$$u(0) = \oint_{\mathbb{T}} \frac{1}{2} \nu(e^{it}) dt = \frac{1}{2} \widehat{\nu}(0)$$

which means that $C = 0$. Therefore, we can define q as the radial limit of $z \mapsto \exp(F(z))$ which coincides with

$$q(t) = \exp\left(\frac{1}{2}\nu(e^{it}) + i/2\mathcal{H}_0\nu(e^{it})\right) = \sqrt{f(t)} \exp\left(i/2\mathcal{H}_0(\log f(t))\right)$$

□

The Lemma 5.2 provides the analytic square root for the reciprocal of the symbol g_α and it follows from Lemma 5.1.

Proof of Lemma 5.2. We know by Lemma 5.1 that q_α is of form

$$t \mapsto \rho_1(e^{it}) \exp\left(-\frac{1}{2}(I + i\mathcal{H}_0) \log g_\alpha(t)\right)$$

for some inner function ρ_1 . Since $w_\alpha \overline{w_\alpha} = \theta_{2\alpha}$, we know by Lemma 5.1 that

$$w_\alpha(t) = \rho(e^{it}) \exp\left(\frac{1}{2}(I + i\mathcal{H}_0) \log \theta_{2\alpha}\right)$$

where ρ is an inner function. We define q_α by choosing $\rho_1 = \rho$. This means that

$$q_\alpha(t)/w_\alpha(t) = \exp\left(\frac{1}{2}(I + i\mathcal{H}_0)\log(\theta_{2\alpha}(t)g_\alpha^{-1}(t))\right) = r_\alpha.$$

□

The Lemma 5.4 gives the asymptotics of the Fourier coefficients of ψ_α and it also gives the asymptotics of the related functions via the mapping properties of Hilbert transform.

Proof of Lemma 5.4. We notice that

$$\theta_{2\alpha}(t) = |t|^{2\alpha} (1 + c_1 t^2 + t^4 \varphi_1(t))$$

for a certain $\varphi_1 \in C^\infty$ and

$$g_\alpha(t) = |t|^{-2\alpha} \varphi_{2,\alpha}(t) + t^2 \varphi_{3,\alpha}(t)$$

for a certain $C^\infty(\mathbb{T})$ function $\varphi_{3,\alpha}$ such that $\varphi_{3,\alpha}(0) > 0$ and where on the cut-off function $\varphi_{2,\alpha} \in C^\infty(\mathbb{T})$ with support in $(-1, 1)$ and $\varphi_2(t) = 1$ for $t \in (-\frac{1}{2}, \frac{1}{2})$. This implies that $u = \log(\theta_{2\alpha}g_\alpha)$ is a $C^1(\mathbb{T})$ -function and the second weak derivative is in $L^1(\mathbb{T})$ and moreover,

$$u''(t) = c_\alpha |t|^{2\alpha} \varphi_{3,\alpha}(t) + u_2(t) = c_\alpha \varphi_{3,\alpha}(0) g_{-\alpha}(t) + u_3(t)$$

where $c_\alpha = (2\alpha + 2)(2\alpha + 1)$ and u_2 and u_3 are certain $C(\mathbb{T})$ -functions with integrable derivative. This implies that $|\widehat{u}_3(k)| \lesssim k^{-1}$ for every $\alpha \in (-\frac{1}{2}, \frac{1}{2})$.

Moreover, since $\widehat{g}_\alpha(k) \asymp c_{2,\alpha} k^{2\alpha-1}$, we have already shown that $\widehat{u} = c_{3,\alpha} k^{-3-2\alpha} + o(k^{-3-2\alpha})$ in the case when $\alpha < 0$. If $\alpha > 0$ we need to first differentiate u_3 and since

$$u'_3(t) = c_{4,\alpha} |t|^{2\alpha} \varphi'_{3,\alpha}(t) + u_4(t)$$

we deduce that $\widehat{u}_3 \in \mathcal{O}k^{-2}$ and the $\widehat{u} = c_{3,\alpha} k^{-3-2\alpha} + o(k^{-3-2\alpha})$ holds for $\alpha \in (0, \frac{1}{2})$ as well. If we continue differentiation and removal of the leading singularity, we obtain a full asymptotic expansion of \widehat{u} .

From the existence of the asymptotic expansion of \widehat{u} , we see that $\widehat{w}(k) := \widehat{u}(k-1) - 2\widehat{u}(k) + \widehat{u}(k+1) \asymp c_{5,\alpha} k^{-5-2\alpha}$ where $w(t) = (1 - \cos t)u(t)$. This implies that $(1 - \cos t)\mathcal{H}_0 u$ is smoother than $\mathcal{H}_0 u$ and therefore, the zero is the only point that gives a contribution to the Fourier series of $v := \frac{1}{2}(I + \mathcal{H}_0)u$ and thus,

$$\widehat{r}(k) = e^v(0)v(0)^{-1}k^{-2}\widehat{v}(k) \asymp k^{-3-2\alpha}.$$

Another but more complicated way is to apply Bojanic–Karamata Tauberian Theorem ([6, Theorem 4.3.2]) to deduce this fact. □

In Lemma 5.5 we compute the asymptotics of the analytic square root of the pure Fisher–Hartwig symbol using the explicit representation of w_α .

Proof of Lemma 5.5. Since $w_\alpha(t) = (1 - e^{it})^\alpha$ and since

$$(1 - z)^\alpha = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} z^k$$

we can deduce that

$$\widehat{w}_\alpha(k) = (-1)^k \binom{\alpha}{k} = (-1)^k \frac{\alpha^{\underline{k}}}{k!} = \frac{(k-1-\alpha)^{\underline{k}}}{k!} = -\alpha \prod_{j=2}^k \left(1 - \frac{\alpha+1}{j}\right)$$

Since $\alpha+1 \in (\frac{1}{2}, \frac{3}{2})$, it holds that $|j^{-1}(\alpha+1)| < 1$ for every $j \geq 2$ and so

$$\log(-\alpha^{-1} \widehat{w}_\alpha(k)) = \sum_{j=2}^k \log(1 - \frac{\beta}{j}) = -\beta \sum_{j=2}^k j^{-1} - \sum_{j=2}^k \sum_{n=2}^{\infty} \frac{\beta^n}{n j^n}$$

Therefore,

$$\log(-\alpha^{-1} \widehat{w}_\alpha(k)) = -\beta \int_1^k t^{-1} dt + C_\beta + \mathcal{O}k^{-1} = \log k^{-\beta} + C_\beta + \mathcal{O}k^{-1}$$

for some constant C_β . However, since

$$\widehat{w}_\alpha(k) = (-1)^k \binom{\alpha}{k} = (-1)^k \frac{\alpha^{\underline{k}}}{k!} = \frac{(k-\beta)^{\underline{k}}}{k!} = k^{-\beta} \left(\frac{k! k^{-\beta}}{(k-\beta)^{\underline{k}}} \right)^{-1}$$

and by Gauss limit formula for Gamma function, we have

$$\lim_{k \rightarrow \infty} \widehat{w}_\alpha(k) k^\beta = \lim_{k \rightarrow \infty} \left(\frac{k! k^{-\beta}}{(k-\beta)^{\underline{k}}} \right)^{-1} = \Gamma(-\beta)^{-1}$$

which is valid every $-\beta \notin \mathbb{Z}$. Therefore,

$$\widehat{w}_\alpha(k) = \Gamma(-\beta)^{-1} k^{-\beta} (1 + \mathcal{O}k^{-1})$$

and the claim follows. \square

The last piece (Lemma 5.6) combines the previous lemmata with a straightforward convolution argument.

Proof of Lemma 5.6. Since the convolution of Fourier transforms is the Fourier transform of the product, we notice that claim is equivalent with

$$\widehat{w_\alpha \nu}(k) = \mathcal{O}k^{-2-\alpha}$$

where $\nu(t) = r_\alpha(t) - r_\alpha(0)$. Furthermore,

$$\widehat{\nu}(k) = [k \neq 0] \widehat{r}_\alpha(k)$$

and so

$$|\widehat{w}_\alpha * \widehat{\nu}(k)| = \left| \sum_{j=1}^k \widehat{w}_\alpha(k-j) \widehat{r}_\alpha(j) \right| \lesssim k^{-3-2\alpha} + \sum_{j=1}^{k-1} (k-j)^{-1-\alpha} j^{-3-2\alpha}.$$

The last sum can be estimated with an integral

$$\int_1^{k-1} (k-t)^{-\beta} t^{-1-2\beta} dt = k^{-3\beta} \int_h^{1-h} (1-s)^{-\beta} s^{-1-2\beta} ds$$

where $\beta = \alpha + 1$. The integral on the right is integrable at zero for every $\beta \in (\frac{1}{2}, \frac{3}{2})$ but it is integrable at one only when $\beta \in (\frac{1}{2}, 1)$. Therefore, when $\beta \in (\frac{1}{2}, 1)$ we have an estimate

$$|\widehat{w}_\alpha * \widehat{\nu}(k)| \lesssim k^{-3-2\alpha} + k^{-3\beta} \asymp k^{-3-3\alpha} \leq k^{-2-\alpha}$$

for $k > 0$, since $-1 - 2\alpha \leq 0$. When $\beta \in (1, \frac{3}{2})$ we estimate

$$\int_h^{1-h} (1-s)^{-\beta} s^{-1-2\beta} ds \asymp \int_{\frac{1}{2}}^{1-h} (1-s)^{-\beta} ds \asymp h^{1-\beta} = k^{\beta-1}$$

and therefore,

$$|\widehat{w}_\alpha * \widehat{\nu}(k)| \lesssim k^{-3-2\alpha} + k^{-1-2\beta} \asymp k^{-3-2\alpha} \leq k^{-3} \leq k^{-2-\alpha}$$

since now $\alpha > 0$. □

APPENDIX D. PROOFS OF AUXILIARY RESULTS IN SECTION 6

This section is dedicated to the technical results that were postponed in Section 3. We begin with the almost trivial proof of Lemma 6.2.

Proof of Lemma 6.2. This follows from the two observations.

- i) $\partial_\alpha(A_\alpha B_\alpha) = (\partial_\alpha A_\alpha)B_\alpha + A_\alpha \partial_\alpha B_\alpha$ for every differentiable A_α and B_α
- ii) $\partial_\alpha(A_\alpha A_\alpha^{-1}) = 0$ for every invertible and differentiable A_α

Using these and simple algebra the claim follows. □

Next we show the Lemma 6.3 which combines 6.2 with analysis of the symbol g_α .

Proof of Lemma 6.3. The part (1) follows by the fact that the symbols g_α are uniformly bounded from below by $1/\lambda_1 > 0$ when $\alpha > 0$.

The part (2) follows by differentiating the symbol g_α and noticing that $\partial_\alpha g_\alpha$ are outside a neighbourhood of zero uniformly bounded from above by $\lambda_2 > 0$. In the neighbourhood of zero $\partial_\alpha g_\alpha(t) \asymp |t|^{-2\alpha} \log |t|^{-1}$ and we can choose a uniform neighbourhood where this holds.

The part (3) follows from (1), (2) and Lemma 6.2 since

$$\langle \partial_\alpha T_n(g_\alpha)^{-1} z, z \rangle = \langle -\partial_\alpha T_n(g_\alpha) w_\alpha, w_\alpha \rangle \leq 0$$

where $w_\alpha = T_n(g_\alpha)^{-1} z$. □

The proof of Lemma 6.4 is very similar to the proof of Lemma 6.2.

Proof of Lemma 6.4. The part (1) follows by comparing the symbols g_α . At the neighbourhood of the zero the $g_\alpha(t) \asymp t^{2|\alpha|} \geq t^{2|\gamma|} \asymp g_\gamma(t)$. Since outside the origin the symbols are uniformly bounded from above and below, we can choose $\lambda_3 > 0$ so that $g_\alpha \geq 1/\lambda_3 g_\gamma$.

The part (2) follows by analysing the derivative $\partial_\alpha g$ of the symbol. In the neighbourhood of zero the $\partial_\alpha g_\alpha(t) \asymp t^{2|\alpha|} \log |t|^{-1}$ is strictly positive when $t \neq 0$ and zero when $t = 0$. Outside the neighbourhood of zero the functions $\partial_\alpha g_\alpha$ change the sign but are uniformly bounded from above. Therefore, we can choose $\lambda_4 > 0$ such that $\partial_\alpha g_\alpha \geq -\lambda_4 g_\alpha$.

Part (2) uses again Lemma 6.2 and parts (1) and (2). Denoting $w_\alpha = T_n(g_\alpha)^{-1}$ we get

$$\begin{aligned} \langle \partial_\alpha T_n(g_\alpha)^{-1} z, z \rangle &= \langle -\partial_\alpha T_n(g_\alpha) w_\alpha, w_\alpha \rangle \leq \lambda_4 \langle z, w_\alpha \rangle \\ &\leq \lambda_3 \lambda_4 \langle z, T_n(g_\gamma)^{-1} z \rangle \end{aligned}$$

as claimed. \square

Proof of Lemma 6.7. Let us first choose a $\lambda'_0 > 0$ such that $\partial_\alpha g_\alpha(\lambda) \geq 0$ for every $|\lambda| \leq \lambda'_0$ and for every $\alpha < 0$ which is possible since in the neighbourhood of zero the derivative $\partial_\alpha g$ behaves like $t^{-2|\alpha|} \log |t|^{-1}$. After choosing such a $\lambda'_0 > 0$ we can compute $\lambda''_0 > 0$ which is the $\mu'_0 := \inf\{g_\alpha(\lambda) \mid |\lambda| > \lambda'_0, \alpha < 0\}$. If $\sup_\alpha g_\alpha(\lambda'_0) \leq \mu'_0$ we define $\lambda_0 = \lambda'_0$ otherwise we just take $\lambda_0 < \lambda'_0$ so small that $\sup_\alpha g_\alpha(\lambda_0) \leq \mu'_0$ which is possible since the g_α are equicontinuous in the neighbourhood of origin for $\alpha \in [-\frac{1}{2} + \varepsilon, -\varepsilon]$.

The auxiliary symbol $\tilde{g}_\alpha(\lambda) := g_\alpha(\lambda \wedge \lambda_0)$ is now seen to satisfy the conditions for large enough $\lambda_5 > 0$. \square

APPENDIX E. PROOFS OF AUXILIARY RESULTS IN SECTION 7

Proof of Lemma 7.6. We already know that $|\alpha_n - \hat{H}| \leq M/\log n$ and therefore, we can use an ansatz $\alpha_n = \hat{H} + \theta$. Since $n^{-2\theta} = \varphi_n(\alpha_n)$ we can use Taylor expansions for \hat{F} and \hat{F}' around \hat{H} and we get an equation

$$n^{-2\theta} = 1 + \hat{F}'(\hat{H})\theta + \frac{\hat{F}''(\hat{H})}{2\log n}\theta^2 + \mathcal{O}(\theta^3) + \mathcal{O}(\theta(\log n)^{-1}).$$

The apriori estimate $\theta = \mathcal{O}(\log n)^{-1}$ reduces this to

$$(45) \quad n^{-2\theta} = 1 + \hat{F}'(\hat{H})\theta + \frac{\hat{F}''(\hat{H})}{2\log n}\theta^2 + \mathcal{O}(\log n)^{-2}.$$

We can now take logarithms, and use the apriori estimate for terms $\mathcal{O}(\theta^2)$ and $\mathcal{O}(\theta(\log n)^{-1})$. Therefore, the asymptotic representation (45) reduces to an asymptotic linear equation for θ namely,

$$\theta(-2\log n - \hat{F}''(\hat{H})) = \frac{\hat{F}''(\hat{H})}{2\log n} + \mathcal{O}(\log n)^{-2}$$

which proves the claim. \square

Proof of Lemma 7.7. We already have that

$$\kappa_n''(\alpha) = -n^{2(\alpha-\hat{H})+1}(\log n)^2\psi_n(\alpha).$$

We also know that $n^{2(\alpha_n-\hat{H})} = 1/\varphi_n(\alpha_n)$. Therefore,

$$\kappa_n''(\alpha) = -n(\log n)^2 n^{2(\alpha-\alpha_n)} \frac{\psi_n(\alpha)}{\varphi_n(\alpha_n)}.$$

From this we immediately compute that

$$\kappa_n^{(3)}(\alpha) = -n(\log n)^2 n^{2(\alpha-\alpha_n)} \frac{2 \log n \psi_n(\alpha) + \psi_n'(\alpha)}{\varphi_n(\alpha_n)}$$

and

$$\kappa_n^{(4)}(\alpha) = -n(\log n)^2 n^{2(\alpha-\alpha_n)} \frac{4(\log n)^2 \psi_n(\alpha) + 4 \log n \psi_n'(\alpha) + \psi_n''(\alpha)}{\varphi_n(\alpha_n)}.$$

Since

$$\psi_n(\alpha) = 2\varphi_n(\alpha) + \varphi_n'(\alpha)(\log n)^{-1}$$

and for $j \leq 3$ we have

$$\|\varphi_n^{(j)}\|_\infty = \mathcal{O}1$$

it follows that

$$|\kappa_n^{(4)}(\alpha)| \leq 8n(\log n)^4$$

for $|\alpha - \alpha_n| \ll (\log n)^{-1}$. For second and third derivatives we get

$$\kappa_n^{(2)}(\alpha_n) = -2n(\log n)^2(1 + \mathcal{O}(\log n)^{-1})$$

and

$$\kappa_n^{(3)}(\alpha_n) = -4n(\log n)^3(1 + \mathcal{O}(\log n)^{-1})$$

\square

Proof of Lemma 7.8. The remainder part is immediately estimated by

$$J(n) = [\gamma > 0] \frac{e^{-K_n(\hat{F})(\alpha(n))}}{n \log n} \mathcal{O}e^{\gamma n \log n}.$$

where $\gamma := \hat{H} - \frac{1}{2}$.

The lower tail and the upper tail calculations are essentially the same so we only do the lower tail, so we assume that $\hat{V} = (\gamma, \beta_-(n)]$ for some $\beta_-(n) < \alpha(n)$. We re-express the integral

$$\int_\gamma^{\beta_-(n)} e^{\kappa_n(\alpha)} d\alpha = \int_\gamma^{\beta_-(n)} \frac{de^{\kappa_n(\alpha)}}{\kappa_n'(\alpha)}$$

Lemma 7.4 implies that $\kappa_n'(\alpha)$ is monotonically decreasing and thus $1/\kappa_n'$ is a monotonically increasing function and since κ_n' has a unique zero point at α_n , the division is well defined. Therefore,

$$\int_\gamma^{\beta_-(n)} e^{\kappa_n(\alpha)} d\alpha \leq \frac{e^{\kappa_n(\beta_-(n))} - e^{\kappa_n(\gamma)}}{\kappa_n'(\beta_-(n))}$$

This implies the claim. \square

Lemma E.1. *We have that*

$$\kappa'_n(\alpha_n + \theta) = -2\theta n(\log n)^2(1 + o(1))$$

for large enough n and $\theta \ll (\log n)^{-1}$.

Proof of Lemma E.1. By substitution we have

$$\kappa'_n(\alpha_n + \theta) = n \log n (1 - n^{2(\alpha_n - \hat{H})} n^{2\theta} \psi_n(\theta))$$

where $\psi_n(\theta) = \varphi_n(\alpha_n + \theta)$. Since $\kappa'_n(\alpha_n) = 0$ we have by substitution that

$$1 = n^{2(\alpha_n - \hat{H})} \psi_n(0).$$

Therefore,

$$\kappa'_n(\alpha_n + \theta) = n \log n (1 - n^{2\theta} \mu_n(\theta))$$

where $\mu_n(\theta) := \psi_n(\theta)/\psi_n(0)$. Since $\mu_n(0) = 1$ and μ'_n is

$$\mu'_n(\theta) = \frac{\varphi'_n(\alpha_n + \theta)}{\psi_n(0)} = \frac{\hat{F}'(\alpha_n + \theta)}{\psi_n(0)} + \mathcal{O}(\log n)^{-1}$$

we have by the Taylor expansion that

$$\mu_n(\theta) = 1 + \theta(\eta + \mathcal{O}(\log n)^{-1}) + \mathcal{O}\theta^2$$

where $\eta = (\hat{F}'/\hat{F})(\hat{H})$. Hence, we obtain an representation

$$\kappa'_n(\alpha_n + \theta) = n \log n (1 - (1 + 2\theta \log n + \mathcal{O}\theta^2(\log n)^2) \mu_n(\theta))$$

which simplifies to

$$\kappa'_n(\alpha_n + \theta) = n \log n (-2\theta \log n + \mathcal{O}\theta(\log n)^{-1} + \mathcal{O}\theta^2(\log n)^2)$$

which is equivalent with

$$\kappa'_n(\alpha_n + \theta) = -2\theta n(\log n)^2(1 + \mathcal{O}((\log n)^{-2} + |\theta|(\log n))).$$

\square

Proof of Lemma 7.9. Since $\theta = \varepsilon_n n^{-1/2}(\log n)^{-1} \ll (\log n)^{-1}$ we can use Lemma E.1 and we obtain

$$\kappa'_n(\alpha_n \pm \theta) \asymp \mp \theta n(\log n)^2 = \mp \varepsilon_n n^{1/2} \log n.$$

This together with Lemma 7.8 gives the claim. \square

Proof of Lemma 7.10. We use the Taylor expansion on k_n around 0. This gives

$$k_n(\alpha) = \frac{1}{2} k_n''(0) \alpha^2 + \frac{1}{6} k_n^{(3)}(0) \alpha^3 + \mathcal{O}(k_n^{(4)}(0) \alpha^4).$$

Using the change of variable $\alpha' = \tau_n(\alpha)$ we get that

$$\int_{\beta_-(n)}^{\beta} e^{k_n(\alpha - \alpha(n))} d\alpha = \frac{1}{\log n \sqrt{nc_n}} \int_{\tau_-(n)}^{\tau(\beta)} e^{\psi_n(\alpha)} d\alpha$$

where $\psi_n(\alpha) = k_n(\alpha(\log n)^{-1}n^{-1/2}c_n^{-1/2})$. Combining this with the Taylor expansion we see that

$$\psi_n(\alpha) = -\frac{1}{2}\alpha^2 + \mathcal{O}(n^{-1/2}|\alpha|^3) = -\frac{1}{2}\alpha^2(1 + \mathcal{O}n^{-1/2}\varepsilon_n)$$

where we used the facts that $k_n''(0) = -c_n n(\log n)^2$ and the estimate that $k_n'''(0) \lesssim n(\log n)^3$ and the estimate $|\alpha|^3 \leq c_n |\alpha|^2 \varepsilon_n$.

From this we obtain both the estimate from the above and from the below for the integral of $e \circ \psi_n$ and therefore,

$$\frac{1}{\sqrt{2\pi}} \int_{\beta_-(n)}^{\beta} e^{k_n(\alpha - \alpha(n))} d\alpha = \frac{1}{\log n \sqrt{nc_n}} (\Phi \circ \tau(\beta) - \lambda_-(n)) (1 + \mathcal{O}n^{-1/2}\varepsilon_n).$$

□

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